

GREM-like K processes on trees with infinite depth

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Abstract

We take up the issue of deriving the limit as $n \rightarrow \infty$ of the GREM-like K process on a tree with n levels. Under specific conditions on the parameters of the process, implying the martingality of a modification of the underlying clock process sequences, we obtain infinite level clock processes as nontrivial limits of the finite level clocks, and use them to construct a process on a suitable product space which is then shown to be the limit of the n level K processes as $n \rightarrow \infty$. Some properties of the limiting, infinite level K process are established, like an expression for the asymptotic empirical measure of cylinders, giving information on the prospective equilibrium measure of the infinite level dynamics.

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1 Introduction

The K process on a tree with finitely many levels/finite depth appeared in [1] as the weak limit of trap models on a tree with finitely many levels as the *volume* of the tree diverges. With the appropriate choice of parameters, the trap model on a tree

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with finitely many levels is a phenomenological model for a dynamics of the GREM (Generalized Random Energy Model [2]) at low temperature and under a regime of parameters and time scale such that each hierarchy¹ is *close to equilibrium* under the dynamics. It was introduced in [4]; see also [5]. In this case it is called the GREM-like trap model, and the associated K process is called the GREM-like K process. A Glauber dynamics for the REM (Random Energy Model [3]), namely the *random hopping dynamics*, was studied in [6], and shown to exhibit *aging* on a long time scale far from equilibrium; [7] shows that this dynamics, under a time rescaling where it is close to equilibrium, converges to a (single level) K process [8]. Aging results for a random hopping dynamics for the *p-spin* model were derived in [9]; see also [10]. There are so far no published study that we know of of Glauber dynamics for the GREM, but we expect that a properly defined random hopping dynamics for the GREM with the proper parameters and at the right time scale also converges to the GREM-like K process [11].

Let us briefly recall/describe the trap model on a tree with finitely many levels, the associated K process, as well as the GREM-like versions. Consider a tree \mathbb{T}_n with n levels/generations starting from a root \emptyset at level 0. Level 1 has volume M_1 , and each vertex of level i is connected to M_{i+1} vertices of level $i+1$, $i = 0, \dots, n-1$. The trap model on a tree with n levels is a Markov jump process on the leaves of \mathbb{T}_n , which when at a leaf $x|_n$ waits an exponential time of mean $\gamma_n(x|_n)$ and jumps to another leaf $y|_n$, chosen as follows. From $x|_n$, we go down the tree through the unique path connecting it to \emptyset , flipping coins found at each site along the way. The coin at site $x|i$ on level i has probability $p_i(x|i)$ of turning up heads, independently of all the other coins. We set $p_0(\emptyset) = 0$, and $p_i(x|i) = (1 + M_{i+1}\gamma_i(x|i))^{-1}$, where $\gamma_i(x|i)$, $x|i$ sites of \mathbb{T}_n , $i = 1, \dots, n$, are positive parameters of the model. Let $x|i$ be the first site on the way from $x|_n$ to \emptyset whose coin flip turns up tails. Then $y|_n$ is chosen uniformly among the leaves of \mathbb{T}_n which are descendants of $x|i$. Provided the γ parameters satisfy a summability condition (saying roughly that the sum over the leaves of \mathbb{T}_n of the products of the γ 's over the path from each leaf to \emptyset converges as $M_i \rightarrow \infty$, $i = 1, \dots, n$), then (a suitable representation of) this process converges in distribution as $M_i \rightarrow \infty$, $i = 1, \dots, n$, to a process on $\overline{\mathbb{N}}_*^n$, where $\overline{\mathbb{N}}_* = \mathbb{N}_* \cup \{\infty\}$, and $\mathbb{N}_* = \{1, 2, \dots\}$ is the positive integers. We call the limiting process the K process on an n level tree with set of parameters $\{\gamma_i(x|i), x|i \in \mathbb{N}_*^i, i = 1, \dots, n\}$.

It is the goal of this paper to derive a nontrivial limit of the latter process as $n \rightarrow \infty$ in the case of the GREM-like K process. This process is a K process on a finite level tree, characterized by the following choice of parameters. Let n be the depth/number of levels of the tree. For each $i = 1, \dots, n$ and each $x|_{i-1} \in \mathbb{N}_*^{i-1}$ (with $\mathbb{N}_*^0 = \{\emptyset\}$),

¹Let us recall that the GREM is a hierarchical mean-field spin glass, with an arbitrary fixed finite number of hierarchies.

let $\underline{\gamma}_i(x|_{i-1}) := \{\gamma_i(x|_i), x_i \in \mathbb{N}_*\}$ be a Poisson point process on $(0, \infty)$ with intensity function $c_i t^{-1-\alpha_i}$ in decreasing order, independent of each other, where $x|_i = (x|_{i-1}, x_i)$. The constants c_1, \dots, c_n are positive, and for the moment arbitrary, and we must have $0 < \alpha_1 < \dots < \alpha_n < 1$. As already briefly mentioned above, it arises as the scaling limit of the GREM-like trap model of Sasaki and Nemoto [4]. This is a trap model on \mathbb{T}_n which under a suitable time rescaling, and provided the *volumes* M_1, \dots, M_n satisfy a *fine tuning* condition among themselves, becomes a trap model on \mathbb{T}_n with the following set of parameters. For each $i = 1, \dots, n$ and each $x|_{i-1}$ on the $(i-1)$ -th level of \mathbb{T}_n , the M_i descendants of $x|_i$ of $x|_{i-1}$ on level i are *i.i.d.* random variables in the basin of attraction of an α_i -stable law, suitably scaled (in the usual way, so that when ordered they converge in law to $\underline{\gamma}_i(x|_{i-1})$ as $M_i \rightarrow \infty$, $i = 1, \dots, n$). Detailed definitions and constructions of the K processes mentioned above are provided in the next section.

Let us now roughly explain the main steps and ideas of the derivation of our main result. We will show that each coordinate of a suitable version of the n level GREM-like K process converges in probability. Suppose we are looking at its k -th coordinate. The appropriate k level clock process is a key ingredient; let us call it θ_k^n . It is defined in terms of the composition of $n - k$ single level clock processes, and we want to take its limit as $n \rightarrow \infty$. We will do that by means of a martingale convergence theorem. Since $(\theta_k^n)_n$ is *not* a martingale, we introduce a modification, namely $\tilde{\theta}_k^n$, which is. This modification is obtained by inserting *missing factors* in a certain way *at the end of* θ_k^n . These missing factors, which depend on the random parameters only, are themselves obtained via a martingale convergence theorem (this time the randomness comes from the parameters). In order that these martingale properties hold true and the limits are nontrivial, we need to make a specific choice of the constants c_1, c_2, \dots mentioned above, and require that $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$ sufficiently fast. We then have limits for the modified clocks at all levels, and use them to define an infinite level process. After showing that the modifications introduced in the clocks wash away in the limit, we are in position to argue directly that the infinite level process defined with the limiting modified clocks is the limit of the original n level K process as $n \rightarrow \infty$.

We thus obtain an infinite level dynamics which is the limit as $n \rightarrow \infty$ of the n level GREM-like K process (under the appropriate assumptions on parameters). Combined with the convergence result of [1], by abstract nonsense, we have that this infinite level dynamics arises as the scaling limit of GREM-like trap models as both volume and number of levels diverge (in a way which is however *not* specified by the abstract argument), provided of course that the right conditions are in place. Presumably this is also the case for suitable random hopping dynamics for the GREM under appropriate conditions.

At the closure of this introduction, we outline the organization of the remainder of

this article. In Section 2, we define the GREM-like K process in detail and formulate our main convergence result, namely Theorem 2.7. Section 3 is devoted to auxiliary results. In Section 4 we derive the limit of the clock processes, and in Section 5 we prove Theorem 2.7, obtaining along the way properties of the *infinity set* of the limiting K process (that is, the set of times where any of its coordinates equals ∞). And in the final Section 6, we derive asymptotics for the empirical measure of cylinders of the limiting process, thus shedding light on the prospective equilibrium measure of the infinite level dynamics.

2 Model and main result

We start by defining the K process on a tree with finite depth via a slight adaptation of the construction employed in [1]. Many elements of this construction will be used to define the infinite depth version of this process.

The state space of the K process on a tree with depth k is $\bar{\mathbb{N}}_*^k$, where $\mathbb{N}_* = \{1, 2, \dots\}$ and $\bar{\mathbb{N}}_* = \mathbb{N}_* \cup \{\infty\}$. We denote elements of this space by $x|_k = (x_1, \dots, x_k)$. For notation brevity we will often denote $x|_k = x_1 x_2 \dots x_k$ and also denote $x|_k y$ as the concatenation of $x|_k$ and y , that is $x|_k y = (x_1, \dots, x_k, y) \in \bar{\mathbb{N}}_*^{k+1}$.

It will be useful to visualize $\bar{\mathbb{N}}_*^k$ as the nodes at depth k of a tree, with \emptyset as root and node $x|_k y$ as an offspring of node $x|_k$, Figure 1 illustrates this representation.

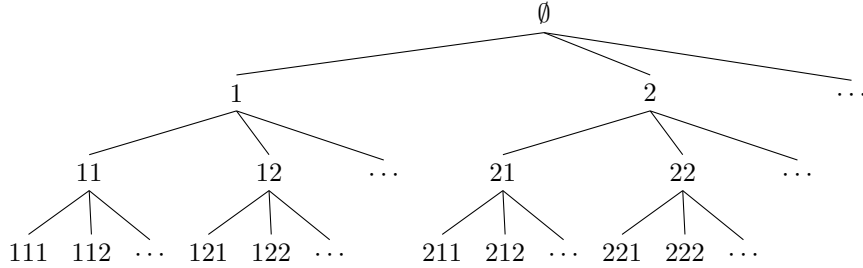


Figure 1: Tree representation of the state space

As parameters, take $0 < \alpha_1 < \alpha_2 < \dots < 1$ real numbers such that $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$.

For each $k \in \mathbb{N}_*$ and $x_{k-1} \in \mathbb{N}_*^{k-1}$ take $\gamma_k(x_{k-1}1) > \gamma_k(x_{k-1}2) > \dots > 0$ the ordered marks of a Poisson point process on \mathbb{R}^+ with intensity measure μ_k :

$$\mu_k(dt) := \frac{c_k}{t^{1+\alpha_k}}, t > 0, \quad c_k := \frac{\alpha_k}{\Gamma\left(1 - \frac{\alpha_k}{\alpha_{k+1}}\right)}, \quad (1)$$

where $\Gamma(\beta) = \int_0^\infty t^{\beta-1} e^{-t} dt$ is the standard Gamma function. This choice for the constant c_k was deliberately made by us in order to obtain convergence of the processes that we are about to construct, as pointed out at the Introduction.

The construction will be made on a probability space that admits all these Poisson processes independently and independent from the following random variables:

- $\{T_i^{k,x} : i, k, x \in \mathbb{N}_*\}$: a family of independent, identically distributed (*i.i.d.*) random variables with exponential distribution of mean 1;
- $\{N^{k,x} : k, x \in \mathbb{N}_*\}$: a family of independent Poisson processes, each with rate 1. We will denote the marks of $N^{k,x}$ by $0 < \sigma_1^{k,x} < \sigma_2^{k,x} < \dots$

We will construct the K processes recursively. Assume $X_0 \equiv \emptyset$ and define for $k = 1, 2, \dots$ and $t \geq 0$:

$$\Xi_k(t) := \sum_{x \in \mathbb{N}_*} \sum_{i=1}^{N^{k,x}(t)} \gamma_k(X_{k-1}(\sigma_i^{k,x})x) T_i^{k,x}, \quad (2a)$$

$$X_k(t) := \begin{cases} (X_{k-1}(\Xi_k^{-1}(t)), x), & \text{if } t \in \bigcup_{i=1}^\infty [\Xi_k(\sigma_i^{k,x}-), \Xi_k(\sigma_i^{k,x})]; \\ (X_{k-1}(\Xi_k^{-1}(t)), \infty), & \text{otherwise,} \end{cases} \quad (2b)$$

where $\Xi_k^{-1}(t) = \inf\{r \geq 0 : \Xi_k(r) > t\}$ is the generalized inverse of Ξ_k .

Definition 2.1. We call X_k the K process on a tree with depth k or with k levels, or still k level K process, with parameter set $\underline{\gamma}_k := \{\gamma_i(x|_i), x|_i \in \mathbb{N}_*^i, i = 1, \dots, k\}$.

Remark 2.1. The processes Ξ_j are the so-called clock processes.

Remark 2.2. Note that our choice for $\{\gamma_k(x|_k) : k \in \mathbb{N}_*, x|_k \in \mathbb{N}_*^k\}$ satisfies almost surely:

$$\sum_{x|_k \in \mathbb{N}_*^k} \bar{\gamma}_k(x|_k) < \infty, \quad \bar{\gamma}_k(x|_k) := \prod_{j=1}^k \gamma_j(x|_j). \quad (3)$$

Any choice for $\{\gamma_k(x|_k)\}$, random or not, that satisfies this condition is suitable for the provided construction, as it guarantees that the clock processes are finite almost surely.

Remark 2.3. This version of the K process, with $\gamma_k(x|_k)$ given randomly as specified above, was called GREM-like K process in [1], where it is shown that, under suitable conditions, it is a scaling limit of the trap models introduced in [4]. In order to prove this result, a slightly different construction is used, which is suited to the coupling argument there undertaken. We opted here for this slightly simpler construction because it will be more adequate to work with further on. Both constructions yield processes with the same law.

Definition 2.2. For $k \leq n$ define the clock composition as:

$$\theta_k^n := \Xi_n \circ \Xi_{n-1} \circ \cdots \circ \Xi_k. \quad (4)$$

For simplicity, these stochastic processes will also be called clocks.

Remark 2.4. Note that, for $j < k$, if $X_{k,j}$ is the j -th coordinate of X_k , the K process on a tree with depth k , then:

$$X_{k,j}(t) = \begin{cases} x, & \text{if } t \in \bigcup_{i=1}^{\infty} [\theta_j^k(\sigma_i^{j,x} -), \theta_j^k(\sigma_i^{j,x})); \\ \infty, & \text{otherwise.} \end{cases} \quad (5)$$

Remark 2.5. We note that, since we did not define $\gamma_k(x|_k)$ for $x|_k \in \overline{\mathbb{N}}_*^k \setminus \mathbb{N}_*^k$, we had better show that $X_{k-1}(\sigma_i^{k,x}) \in \mathbb{N}_*^{k-1}$ a.s..

To do this, we first note that for $j < k$, $X_{k-1,j}(s) = \infty$ if and only if s belongs to the image of θ_j^{k-1} , and this set has null Lebesgue measure, since θ_j^{k-1} is a step function.

Furthermore, we can infer from the construction that a.s., for any $j \leq k$, $X_{k,j}$ is constant and finite in any interval $[\Xi_k(\sigma_i^{k,x} -), \Xi_k(\sigma_i^{k,x}))$.

We also note that if s is a discontinuity point of θ_k^n , for $k \leq n$, then $s = \sigma_i^{k,x}$ for some $i, x \in \mathbb{N}_*$. This can be proven by noting that if s is a discontinuity point of θ_k^n and $s \notin \{\sigma_i^{k,x} : i, x \in \mathbb{N}_*\}$, then $\theta_k^{m-1}(s) = \sigma_j^{m,y}$ for some $k < m \leq n, y \in \mathbb{N}_*$ and therefore $\sigma_j^{m,y}$ belongs to the image of θ_k^{m-1} , event that has probability zero.

Remark 2.4 suggests an approach to define the K processes on a tree with infinite depth by taking the limit of θ_k^n as $n \rightarrow \infty$. In Theorem 4.4 we will define stochastic processes θ_k^∞ , $k \geq 1$, and in Theorem 5.6 we will show that these stochastic processes are in fact the limits of θ_k^n , $k \geq 1$, as $n \rightarrow \infty$. But the nontriviality of the limits requires a condition, namely

$$\sum_{k=1}^{\infty} \frac{1 - \alpha_{k+1}}{1 - \alpha_k} < \infty. \quad (6)$$

We will refer to this condition below as *the nontriviality condition*. In order to define the infinite level K process and state our main result, we will assume for the remaining of this section that these processes θ_k^∞ are already constructed.

Definition 2.3. The K process on a tree with infinite depth or with infinite levels, or still infinite level K process, with parameter set $\underline{\gamma} := \{\gamma_i(x|_i), x|_i \in \mathbb{N}_*^i, i \geq 1\}$, is a continuous time process $\mathbb{Y} = (Y_k)_{k \in \mathbb{N}_*}$ taking values on $\overline{\mathbb{N}}_*^{\mathbb{N}_*}$, where:

$$Y_k(t) = \begin{cases} x, & \text{if } t \in \bigcup_{i=1}^{\infty} [\theta_k^\infty(\sigma_i^{k,x} -), \theta_k^\infty(\sigma_i^{k,x})); \\ \infty, & \text{otherwise.} \end{cases} \quad (7)$$

To be able to state a convergence theorem for the K processes in trees with finite depth to the one on a tree with infinite depth, we need to define an appropriate topology on the state space. For a fixed coordinate we will use a compactification of $\bar{\mathbb{N}}_*$. Namely we equip $\bar{\mathbb{N}}_*$ with the metric ρ_0 :

$$\rho_0(x, y) := \left| \frac{1}{x} - \frac{1}{y} \right|, \quad x, y \in \bar{\mathbb{N}}_*,$$

under the convention that $\frac{1}{\infty} = 0$. In $\bar{\mathbb{N}}_*^{\mathbb{N}}$, we will adopt the metric ρ :

$$\rho(x|_{\infty}, y|_{\infty}) := \sum_{k=1}^{\infty} \frac{\rho_0(x_k, y_k)}{2^k}.$$

Two points $x|_{\infty}$ and $y|_{\infty}$ are close in this metric if they are close on a finite number of coordinates. What happens on “large” coordinates influences little.

We will extend the metric ρ to $\bigcup_{k=1}^{\infty} \bar{\mathbb{N}}_*^k$ by adding a new symbol ζ to the state space and define $\rho_0(\zeta, x) := \mathbb{I}\{x = \zeta\}$. Then extend ρ by “adding ζ at the end” of $x|_k$. That is, for $j \leq k \leq \infty$:

$$\begin{aligned} \rho(x|_j, y|_k) &:= \sum_{i=1}^j \frac{\rho_0(x_i, y_i)}{2^i} + \sum_{i=j+1}^k \frac{\rho_0(\zeta, y_i)}{2^i}, \\ \rho(y|_k, x|_j) &:= \rho(x|_j, y|_k). \end{aligned}$$

Remark 2.6. ρ is a complete metric over $\bar{\mathbb{N}}_*^{\mathbb{N}} \cup \bigcup_{k=1}^{\infty} \bar{\mathbb{N}}_*^k$. It also generates a separable and compact topology.

Theorem 2.7. Under the nontriviality condition (6), \mathbb{Y} is a càdlàg process under ρ and X_k converges to \mathbb{Y} as $k \rightarrow \infty$ in probability under the Skorohod topology using ρ .

Remark 2.8. As we shall see below, conditions (1) and (6) are crucial in our approach. Without them we cannot insure neither the existence nor the nontriviality of the limiting clocks θ_k^{∞} . See Remark 5.14 below.

Remark 2.9. We will denote by \mathbb{P} the underlying probability measure, with \mathbb{E} as expectation. We will use the notation \mathbb{E}_{γ} for the conditional expectation given $\underline{\gamma}$.

3 Preliminaries

In this section we will prove some auxiliary results on $\{\gamma_k(x|_k) : x|_k \in \bigcup_{j=1}^{\infty} \bar{\mathbb{N}}_*^j\}$, which may be thought of as a random environment for the process.

Definition 3.1. For a fixed $k, n \in \mathbb{N}_*$, $k \leq n$, and $x|_k \in \overline{\mathbb{N}}_*^k$ let us denote the “cylinders” based on $x|_k$ as:

$$\begin{aligned} [x|_k]_n &:= \{y|_n \in \mathbb{N}_*^n : y|_k = x|_k\}, \\ \overline{[x|_k]}_n &:= \{y|_n \in \overline{\mathbb{N}}_*^n : y|_k = x|_k\}. \end{aligned}$$

Proposition 3.1. Let $\{\gamma_i : i \in \mathbb{N}\}$ be the marks of a Poisson point process on \mathbb{R}^+ with intensity measure $\mu(dt) = c/t^{1+\alpha}$ for $t > 0$ with some $\alpha \in (0, 1)$ and $c > 0$. Let $\{X_i : i \in \mathbb{N}\}$ be i.i.d. positive random variables, independent from the Poisson process, such that $\mathbb{E}(X_1^\alpha) < \infty$. Then $\{\gamma_i X_i : i \in \mathbb{N}\}$ is also a Poisson Point process, with intensity measure $\mathbb{E}(X_1^\alpha)\mu$.

Proof. Define $S := \{(\gamma_i, X_i) : i \in \mathbb{N}\}$ and note that S is the set of the marks of a Poisson point process on the first quadrant of \mathbb{R}^2 with intensity measure $\pi = \mu \times \nu$, where ν is the probability measure of X_1 .

Note that $T(x, y) = xy$ is a continuous transformation without accumulation points outside of zero, so $\{T(s) : s \in S\} = \{\gamma_i X_i : i \in \mathbb{N}\}$ is a Poisson point process (see eg. [16], Mapping Theorem, Section 2.3). Its intensity measure can be computed as $\mathbb{E}(X_1)\mu$. \square

Proposition 3.2. Let X be a positive random variable, with Laplace transform $\phi(\lambda) := \mathbb{E}(e^{-\lambda X}) = e^{-c\lambda^\alpha}$, for some $c > 0$, $\alpha \in (0, 1)$. Then for $0 < \beta < \alpha$:

$$\mathbb{E}(X^\beta) = c^{\beta/\alpha} \frac{\Gamma(1 - \beta/\alpha)}{\Gamma(1 - \beta)}.$$

Proof. Using Fubini’s theorem and the fact that $\phi'(\lambda) = -\mathbb{E}(X e^{-\lambda X})$ one can readily check that:

$$\mathbb{E}(X^\beta) = -\frac{1}{\Gamma(1 - \beta)} \int_0^\infty \phi'(\lambda) \lambda^{-\beta} d\lambda = c^{\beta/\alpha} \frac{\Gamma(1 - \beta/\alpha)}{\Gamma(1 - \beta)}. \quad \square$$

Proposition 3.3. For fixed $j, k \in \mathbb{N}_*$, $j < k$, $x|_j \in \mathbb{N}_*^j$ and $\lambda > 0$:

$$\mathbb{E} \left[\exp \left\{ -\lambda \sum_{y|_k \in [x|_j]_k} \frac{\bar{\gamma}_k(y|_k)}{\bar{\gamma}_j(y|_j)} \right\} \right] = \exp \left\{ - \left[\frac{\Gamma(1 - \alpha_k)}{\Gamma(1 - \frac{\alpha_k}{\alpha_{k+1}})} \right]^{\alpha_{j+1}/\alpha_k} \lambda^{\alpha_{j+1}} \right\}. \quad (8)$$

Therefore $\sum_{y|_k \in [x|_j]_k} \frac{\bar{\gamma}_k(y|_k)}{\bar{\gamma}_j(y|_j)}$ has an α_{j+1} -stable distribution and thus is finite a.s..

Proof. This computation will be done by induction on j . We omit the base, that is, when $j = k - 1$ because it can be done analogously as the induction step.

Taking $0 \leq j < k - 1$, let us assume that (8) is true for $j + 1$, and show that it is also true for j .

Using Campbell's Theorem (see eg. [16], Section 3.2) and Propositions 3.1 and 3.2, together with the induction hypothesis, we have that:

$$\begin{aligned}
& \mathbb{E} \left[\exp \left\{ -\lambda \sum_{y|_k \in [x|_j]_k} \frac{\bar{\gamma}_k(y|_k)}{\bar{\gamma}_j(y|_j)} \right\} \right] \\
&= \mathbb{E} \left[\exp \left\{ -\lambda \sum_{x_{j+1}} \gamma_{j+1}(x|_{j+1}) \sum_{y|_k \in [x|_{j+1}]_k} \frac{\bar{\gamma}_k(y|_k)}{\bar{\gamma}_j(y|_{j+1})} \right\} \right] \\
&= \exp \left\{ -\int_0^\infty (1 - e^{-\lambda t}) \mathbb{E} \left[\left(\sum_{y|_k \in [x|_{j+1}]_k} \frac{\bar{\gamma}_k(y|_k)}{\bar{\gamma}_j(y|_{j+1})} \right)^{\alpha_{j+1}} \right] \mu_{j+1}(dt) \right\} \\
&= \exp \left\{ -\left[\frac{\Gamma(1 - \alpha_k)}{\Gamma\left(1 - \frac{\alpha_k}{\alpha_{k+1}}\right)} \right]^{\alpha_{j+1}/\alpha_k} \lambda^{\alpha_{j+1}} \right\}. \quad \square
\end{aligned}$$

We are interested on the random variables treated on Proposition 3.3 and would like to be able to take limits as $k \rightarrow \infty$. But we are unable to do so directly. Instead we have the following proposition:

Proposition 3.4. *For every $x|_k \in \mathbb{N}_*^k$, the following limit exists almost surely:*

$$W(x|_k) := \lim_{n \rightarrow \infty} \sum_{y|_n \in [x|_k]_n} \left(\frac{\bar{\gamma}_n(y|_n)}{\bar{\gamma}_k(y|_k)} \right)^{\alpha_{n+1}}. \quad (9)$$

Moreover $W(x|_k)$ has an α_{k+1} -stable distribution with Laplace transform:

$$\mathbb{E} [e^{-\lambda W(x|_k)}] = \exp\{-\lambda^{\alpha_{k+1}}\}. \quad (10)$$

Proof. Without loss of generality, let us assume $k = 0$ and define random variables $Z_n(\lambda)$ for $n > k$ and $\lambda > 0$ as:

$$Z_n(\lambda) := \exp \left\{ -\sum_{y|_n} (\lambda \bar{\gamma}_n(y|_n))^{\alpha_{n+1}} \right\}.$$

Let us show that these variables are a martingale in relation to the filtration (\mathcal{D}_n) , where \mathcal{D}_n is the σ -algebra generated by all Poisson point processes $\{\gamma_j(x|_j) : j \leq n, x|_j \in \mathbb{N}_*^j\}$. This is done again using Campbell's Theorem:

$$\begin{aligned}
\mathbb{E}(Z_{n+1}(\lambda)|\mathcal{F}_n) &= \mathbb{E} \left[\exp \left\{ - \sum_{y|_{n+1}} (\lambda \bar{\gamma}_{n+1}(y|_{n+1}))^{\alpha_{n+2}} \right\} \middle| \mathcal{D}_n \right] \\
&= \prod_{y|_n} \mathbb{E} \left[\exp \left\{ - (\lambda \bar{\gamma}_n(y|_n))^{\alpha_{n+2}} \sum_{y_{n+1}} \gamma_{n+1}(y|_{n+1})^{\alpha_{n+2}} \right\} \middle| \mathcal{D}_n \right] \\
&= \prod_{y|_n} \exp \left\{ - \int_0^\infty (1 - \exp(-(\lambda \bar{\gamma}_n(y|_n))^{\alpha_{n+2}} t^{\alpha_{n+2}})) \mu_{n+1}(dt) \right\} \\
&= \prod_{y|_n} \exp \{ - (\lambda \bar{\gamma}_n(y|_n))^{\alpha_{n+1}} \} \\
&= \exp \left\{ - \sum_{y|_n} (\lambda \bar{\gamma}_n(y|_n))^{\alpha_{n+1}} \right\} = Z_n(\lambda)
\end{aligned}$$

Since $(Z_n(1))$ is a positive martingale, using a martingale convergence theorem (see eg. [12], Theorem 5.2.9) we conclude that $Z_n(1)$ converges *a.s.*. Since $Z_n(1)$ converges, then its exponent must converge as well.

The second claim can be obtained by using the previous result to explicitly compute:

$$\begin{aligned}
\mathbb{E} \left[\exp \left\{ - \lambda \sum_{y|_n} \bar{\gamma}_n(y|_n)^{\alpha_{n+1}} \right\} \right] &= \mathbb{E} [Z_n(\lambda^{1/\alpha_{n+1}})] \\
&= \mathbb{E} [Z_1(\lambda^{1/\alpha_{n+1}})] \\
&= \exp \{ - \lambda^{\alpha_1/\alpha_{n+1}} \} \xrightarrow{n \rightarrow \infty} \exp \{ - \lambda^{\alpha_1} \}. \quad \square
\end{aligned} \tag{11}$$

Proposition 3.5. *The family of random variables $\{W(x|_k) : x|_k \in \bigcup_{j=0}^\infty \mathbb{N}_*^j\}$ satisfies a composition law. Namely for every $x|_k \in \mathbb{N}_*^k$:*

$$W(x|_k) = \sum_{x_{k+1}} \gamma_{k+1}(x|_{k+1}) W(x|_{k+1}) \text{ a.s.} \tag{12}$$

Proof. Fix a realization such that (9) is true for every $x|_k \in \bigcup_{j=1}^\infty \mathbb{N}_*^j$ and fix an arbitrary

$\epsilon > 0$, then:

$$\begin{aligned}
W(x|_k) &= \lim_{n \rightarrow \infty} \sum_{y_n \in [x|_k]_n} \left(\frac{\bar{\gamma}_n(y|_n)}{\bar{\gamma}_k(x|_k)} \right)^{\alpha_{n+1}} \\
&= \lim_{n \rightarrow \infty} \sum_{x_{k+1}} (\gamma_{k+1}(x|_{k+1}))^{\alpha_{n+1}} \sum_{y_n \in [x|_{k+1}]_n} \left(\frac{\bar{\gamma}_n(y|_n)}{\bar{\gamma}_{k+1}(x|_{k+1})} \right)^{\alpha_{n+1}} \\
&= \lim_{n \rightarrow \infty} \sum_{\substack{x_{k+1}: \\ \gamma_{k+1}(x|_{k+1}) > \epsilon}} (\gamma_{k+1}(x|_{k+1}))^{\alpha_{n+1}} \sum_{y_n \in [x|_{k+1}]_n} \left(\frac{\bar{\gamma}_n(y|_n)}{\bar{\gamma}_{k+1}(x|_{k+1})} \right)^{\alpha_{n+1}} \quad (13)
\end{aligned}$$

$$+ \lim_{n \rightarrow \infty} \sum_{\substack{x_{k+1}: \\ \gamma_{k+1}(x|_{k+1}) \leq \epsilon}} (\gamma_{k+1}(x|_{k+1}))^{\alpha_{n+1}} \sum_{y_n \in [x|_{k+1}]_n} \left(\frac{\bar{\gamma}_n(y|_n)}{\bar{\gamma}_{k+1}(x|_{k+1})} \right)^{\alpha_{n+1}}. \quad (14)$$

Let us treat terms these two terms separately and show that their respective limits exist in probability. For (13), since the outermost sum is finite and $\alpha_n \rightarrow 1$ as $n \rightarrow \infty$, we have that:

$$\begin{aligned}
(13) &= \sum_{\substack{x_{k+1}: \\ \gamma_{k+1}(x|_{k+1}) > \epsilon}} \gamma_{k+1}(x|_{k+1}) \lim_{n \rightarrow \infty} \sum_{y_n \in [x|_{k+1}]_n} \left(\frac{\bar{\gamma}_n(y|_n)}{\bar{\gamma}_k(x|_{k+1})} \right)^{\alpha_{n+1}} \\
&= \sum_{\substack{x_{k+1}: \\ \gamma_{k+1}(x|_{k+1}) > \epsilon}} \gamma_{k+1}(x|_{k+1}) W(x|_{k+1}) \xrightarrow{\epsilon \rightarrow 0} \sum_{x_{k+1}} \gamma_{k+1}(x|_{k+1}) W(x|_{k+1}).
\end{aligned}$$

Let $\mathcal{D}'_k = \mathcal{D}'_k(x|_k)$ be the σ -algebra generated by $\{\gamma_{k+1}(x|_{k+1}) : x_{k+1} \in \mathbb{N}_*\}$. Using

(11), we can compute the Laplace transform of (14) as:

$$\begin{aligned}
& \mathbb{E} \left[\exp \left\{ -\lambda \sum_{\substack{x_{k+1}: \\ \gamma_{k+1}(x|_{k+1}) \leq \epsilon}} (\gamma_{k+1}(x|_{k+1}))^{\alpha_{n+1}} \sum_{y_n \in [x|_{k+1}]_n} \left(\frac{\bar{\gamma}_n(y|_n)}{\bar{\gamma}_{k+1}(x|_{k+1})} \right)^{\alpha_{n+1}} \right\} \right] \\
&= \mathbb{E} \left[\prod_{\substack{x_{k+1}: \\ \gamma_{k+1}(x|_{k+1}) \leq \epsilon}} \mathbb{E} \left[\exp \left\{ -\lambda (\gamma_{k+1}(x|_{k+1}))^{\alpha_{n+1}} \sum_{y_n \in [x|_{k+1}]_n} \left(\frac{\bar{\gamma}_n(y|_n)}{\bar{\gamma}_{k+1}(x|_{k+1})} \right)^{\alpha_{n+1}} \right\} \middle| \mathcal{D}'_k \right] \right] \\
&= \mathbb{E} \left[\prod_{\substack{x_{k+1}: \\ \gamma_{k+1}(x|_{k+1}) \leq \epsilon}} \exp \left\{ -(\lambda (\gamma_{k+1}(x|_{k+1}))^{\alpha_{n+1}})^{\frac{\alpha_{k+2}}{\alpha_{n+1}}} \right\} \right] \\
&\xrightarrow{n \rightarrow \infty} \mathbb{E} \left[\exp \left\{ -\lambda^{\alpha_{k+2}} \sum_{\substack{x_{k+1}: \\ \gamma_{k+1}(x|_{k+1}) \leq \epsilon}} (\gamma_{k+1}(x|_{k+1}))^{\alpha_{k+2}} \right\} \right] \\
&= \exp \left\{ -\int_0^\epsilon \left(1 - e^{-(\lambda x)^{\alpha_{k+2}}} \right) \mu_{k+1}(dx) \right\} \xrightarrow{\epsilon \rightarrow 0} 1.
\end{aligned}$$

We used Campbell's Theorem for the last passage. With this we can conclude that the quantity in (14) converges in probability to 0 as $\epsilon \rightarrow 0$.

Finally, note that we have shown that (13) + (14) converge in probability to both $W(x|_k)$ and $\sum_{x_{k+1}} \gamma_{k+1}(x|_{k+1}) W(x|_{k+1})$ as $\epsilon \rightarrow 0$, which implies that these last quantities are almost surely equal. \square

4 Limiting clocks

Our objective in this section is to define the limiting clocks θ_k^∞ mentioned in Section 2. For this purpose we will introduce a perturbation to the clocks using the variables $W(x|_k)$ introduced in Proposition 3.4. Theorem 5.6 of the next section will prove that this perturbation does not affect the limit.

Definition 4.1. *We define the adjusted clocks as:*

$$\begin{aligned}
\tilde{\Xi}_j(t) &:= \sum_{x \in \mathbb{N}_*} \sum_{i=1}^{N^{j,x}(t)} W(X_{j-1}(\sigma_i^{j,x})x) \gamma_j(X_{j-1}(\sigma_i^{j,x})x) T_i^{j,x} \\
\tilde{\theta}_k^n &:= \tilde{\Xi}_n \circ \Xi_{n-1} \circ \dots \circ \Xi_k.
\end{aligned} \tag{15}$$

Let us also denote the time spent by the K process X_k on a state $x|_k$ up to time t as $L_k(x|_k, t)$, that is:

$$L_k(x|_k, t) = \int_0^t \mathbb{I}\{X_k(s) = x|_k\} ds. \quad (16)$$

For almost every fixed environment $\{\gamma_k(x|_k) : k \in \mathbb{N}_*, x|_k \in \mathbb{N}_*^k\}$, we have that θ_1^n is a subordinator (we will prove this on Lemma 5.4), but θ_k^n , for $k > 1$, is not. This creates some complications. To circumvent most of these problems we will break θ_k^n into a sum of functions that are subordinators.

Definition 4.2. For fixed $n > k \geq 1$, let ν_{k+1}^n be the random measure on the Borel sets of $[0, \infty)$ such that $\nu_{k+1}^n([0, t]) = \theta_{k+1}^n(t)$. For a fixed $x|_k \in \mathbb{N}_*^k$, we define:

$$\theta_{x|_k}^n(t) := \nu_{k+1}^n(\{s \in [0, \infty) : L(x|_k, s) \leq t, X_k(s) = x|_k\}).$$

Figure 2 illustrates this definition. The intervals marked on the abscissa of Figure 2a are the ones where $X_k = x|_k$.

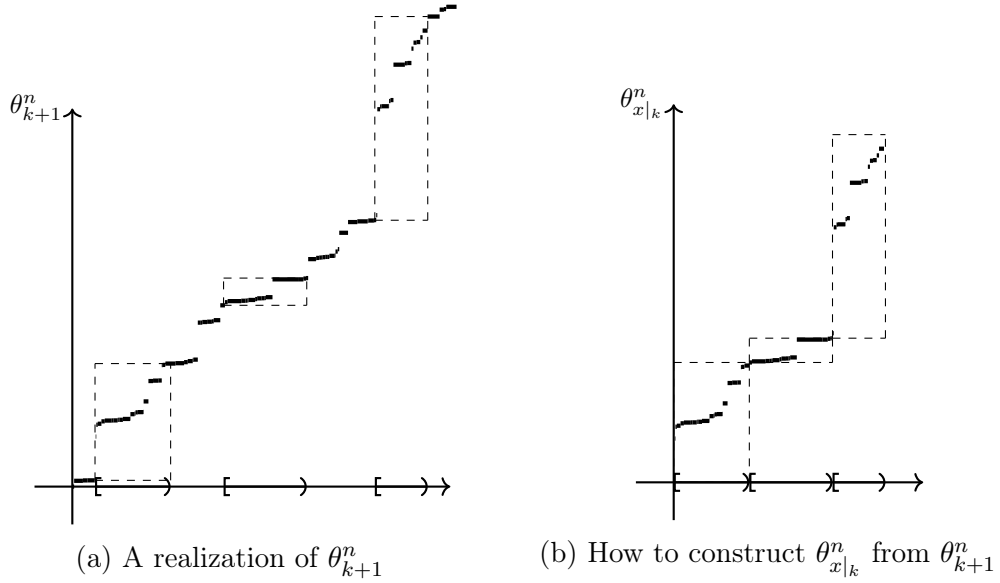


Figure 2: Construction of $\theta_{x|_k}^n$

Remark 4.1. Note that $\theta_{x|_k}^n$ has the same law as θ_1^{n-k} , but with the index of the α 's shifted, that is, it has the same law as a process $\hat{\theta}_1^{n-k}$, constructed the same way as θ_1^{n-k} , but with parameters $(\hat{\alpha}_i)$, where $\hat{\alpha}_i = \alpha_{i+k}$.

These processes are also independent for a fixed n and varying $x|_k$. And it is true that:

$$\theta_{k+1}^n(t) = \sum_{x|_k} \theta_{x|_k}^n(L_k(x|_k, t)) \text{ a.s.} \quad (17)$$

Proposition 4.2. For $n > k \geq 0$, the Laplace transforms of $\theta_{k+1}^n(t)$ and $\tilde{\theta}_{k+1}^n(t)$ satisfy

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ -\lambda \theta_{k+1}^n(t) \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ - \sum_{x|_k \in \mathbb{N}_*^k} L_k(x|_k, t) \sum_{x_{k+1}} h_{x|_{k+1}} \left(\cdots \sum_{x_n} h_{x|_n}(\lambda) \cdots \right) \right\} \right], \end{aligned} \quad (18)$$

$$\begin{aligned} & \mathbb{E} \left[\exp \left\{ -\lambda \tilde{\theta}_{k+1}^n(t) \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ - \sum_{x|_k \in \mathbb{N}_*^k} L_k(x|_k, t) \sum_{x_{k+1}} h_{x|_{k+1}} \left(\cdots \sum_{x_n} h_{x|_n}(\lambda W(x|_n)) \cdots \right) \right\} \right], \end{aligned} \quad (19)$$

where

$$h_{x|_k}(\lambda) := \frac{\lambda \gamma_k(x|_k)}{1 + \lambda \gamma_k(x|_k)},$$

with the convention that $\mathbb{N}_*^0 = \{\emptyset\}$ and $L_0(\emptyset, t) = t$.

Proof. Let \mathcal{E}_n be the σ -algebra generated by all variables involved on the construction up to level n and take $\{Z_{x|_n} : x|_n \in \mathbb{N}_*^n\}$ an arbitrary family of positive independent random variables, this family will be assumed independent of \mathcal{E}_n .

Let us start by proving that:

$$\mathbb{E} \left[\exp \left\{ - \sum_{x|_n} Z_{x|_n} L_n(x|_n, \theta_1^n(t)) \right\} \right] = \mathbb{E} \left[\exp \left\{ -t \sum_{x_1} h_{x|_1} \left(\cdots \sum_{x_n} h_{x|_n}(Z_{x|_n}) \cdots \right) \right\} \right]. \quad (20)$$

The case $k = 0$ will follow by taking $Z_{x|_n} = \lambda$ and $Z_{x|_n} = \lambda W(x|_n)$ for θ_1^n and $\tilde{\theta}_1^n$ respectively.

Note that, because of the way that the K process was constructed in (2b), and the

fact that the increments of a Poisson Process are independent and stationary, then:

$$\begin{aligned} L_n(x|_n, \Xi_n(t)) &= \sum_{i=1}^{N^{n,x_n}(t)} \mathbb{I}\{X_{n-1}(\sigma_i^{n,x_n}) = x|_{n-1}\} \gamma_n(x|_n) T_i^{n,x_n} \\ &\stackrel{\mathcal{D}}{=} \sum_{i=1}^{N^{n,x_n}(L_{n-1}(x|_{n-1}, t))} \gamma_n(x|_n) T_i^{n,x_n} \end{aligned}$$

Letting $\mathcal{F}_n := \sigma(\mathcal{E}_{n-1}, Z_{x|_n} : x|_n \in \mathbb{N}_*^n)$, we can compute:

$$\begin{aligned} \mathbb{E} [e^{-Z_{x|_n} L_n(x|_n, \theta_1^n(t))} | \mathcal{F}_n] &= \mathbb{E} \left[\exp \left\{ -Z_{x|_n} \sum_{i=1}^{N^{n,x_n}(L_{n-1}(x|_{n-1}, \theta_1^{n-1}(t)))} \gamma_n(x|_n) T_i^{n,x_n} \right\} \middle| \mathcal{F}_n \right] \\ &= \exp \{ -L_{n-1}(x|_{n-1}, \theta_1^{n-1}(t)) h_{x|_n}(Z_{x|_n}) \} \end{aligned}$$

Now let us return to (20). We will prove that equality by induction on n . The base, $n = 1$, is obtained by simple inspection. Assuming the result to be true for $n - 1$, we can write:

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ - \sum_{x|_n} Z_{x|_n} L_n(x|_n, \theta_1^n(t)) \right\} \right] &= \mathbb{E} \left[\mathbb{E} \left[\exp \left\{ - \sum_{x|_n} Z_{x|_n} L_n(x|_n, \theta_1^n(t)) \right\} \middle| \mathcal{F}_n \right] \right] \\ &= \mathbb{E} \left[\prod_{x|_n} \mathbb{E} [\exp \{ -Z_{x|_n} L_n(x|_n, \theta_1^n(t)) \} | \mathcal{F}_n] \right] \\ &= \mathbb{E} \left[\exp \left\{ - \sum_{x|_{n-1}} L_n(x|_{n-1}, \theta_1^{n-1}(t)) \sum_{x_n} h_{x|_n}(Z_{x|_n}) \right\} \right]. \end{aligned}$$

Taking $Z_{x|_{n-1}} := \sum_{x_n} h_{x|_n}(Z_{x|_n})$, we can apply the induction hypothesis, obtaining (20), from which we conclude the case $k = 0$.

For the general case, using Remark 4.1, we can write

$$\begin{aligned} \mathbb{E} [e^{-\lambda \theta_{k+1}^n(t)}] &= \mathbb{E} \left[\exp \left\{ -\lambda \sum_{x|_k} \theta_{x|_k}^n(L_k(x|_k, t)) \right\} \right] \\ &= \mathbb{E} \left[\prod_{x|_k} \mathbb{E} [\exp \{ -\lambda \theta_{x|_k}^n(L_k(x|_k, t)) \} | \mathcal{E}_k] \right] \\ &= \mathbb{E} \left[\exp \left\{ - \sum_{x|_k} L_k(x|_k, t) Z_{x|_k}^n \right\} \right]. \end{aligned}$$

The corresponding result to $\tilde{\theta}_{k+1}^n$ can be proven analogously. \square

Remark 4.3. For every $j \leq k$, note that the set $\{t \geq 0 : \mathbb{P}(X_{k,j}(t) = \infty) > 0\}$ has null Lebesgue measure almost surely. This is a consequence of Fubini's Theorem and Remark 2.5:

$$\int_0^\infty \mathbb{P}(X_{k,j}(t) = \infty) dt = \mathbb{E} \left[\int_0^\infty \mathbb{I}\{X_{k,j}(t) = \infty\} dt \right] = 0.$$

Theorem 4.4. For every $k \in \mathbb{N}_*$ there exists almost surely a right-continuous non decreasing process θ_k^∞ such that $\lim_{n \rightarrow \infty} \tilde{\theta}_k^n(t) = \theta_k^\infty(t)$ a.s., $t \in D_k$, where D_k is a countable, deterministic and dense set of $[0, \infty)$. Furthermore, for almost every $\underline{\gamma}$, we have

$$\mathbb{E}_\gamma(\theta_{x|_k}^\infty(t)) \leq tW(x|_k). \quad (21)$$

Proof. Let us fix $\underline{\gamma}$ and prove the result for almost all such choices. We also will denote by \mathcal{G}_k the σ -algebra generated by all Poisson processes and exponential random variables up to level k .

Looking at definition (15), we can split the ranges in which $\tilde{\Xi}_{k+1}$ uses the random environment from each of $x|_k$ of the previous levels, obtaining:

$$\begin{aligned} \mathbb{E}_\gamma \left[\tilde{\Xi}_{k+1}(t) \middle| \mathcal{G}_k \right] &= \sum_{x|_k} L_k(x|_k, t) \sum_{x_{k+1}} W(x|_{k+1}) \gamma_{k+1}(x|_{k+1}) \\ &= \sum_{x|_k} L_k(x|_k, t) W(x|_k). \end{aligned}$$

We can rewrite the definition in (15) to obtain:

$$\begin{aligned} \tilde{\Xi}_n(t) &= \sum_{x_n \in \mathbb{N}_*} \sum_{i=1}^{N^{j, x_n}(t)} W(X_{n-1}(\sigma_i^{n, x_n})x_n) \gamma_n(X_{n-1}(\sigma_i^{n, x_n})x_n) T_i^{n, x_n} \\ &= \sum_{x|_n \in \mathbb{N}_*^n} \sum_{i=1}^{N^{j, x_n}(t)} W(x|_n) \gamma_n(x|_n) \mathbb{I}\{X_{n-1}(\sigma_i^{n, x_n}) = x|_{n-1}\} T_i^{n, x_n} \\ &= \sum_{x|_n \in \mathbb{N}_*^n} W(x|_n) L_n(x|_n, \Xi_n(t)) \end{aligned}$$

Finally taking $n > k$:

$$\begin{aligned} \mathbb{E}_\gamma \left[\tilde{\theta}_k^{n+1}(t) \middle| \mathcal{G}_n \right] &= \mathbb{E}_\gamma \left[\tilde{\Xi}_{n+1}(\theta_k^n(t)) \middle| \mathcal{G}_n \right] \\ &= \sum_{x|_n} L_n(x|_n, \theta_k^n(t)) W(x|_n) \\ &= \tilde{\theta}_k^n(t) \end{aligned}$$

We have thus shown that, under \mathbb{P}_γ , $(\tilde{\theta}_k^n(t))_{n \geq k}$ is a martingale with respect to the filtration $(\mathcal{G}_n)_n$. Since it is nonnegative, we can again use a martingale convergence theorem (see eg. [12], Theorem 5.2.9) to conclude that $\lim_{n \rightarrow \infty} \tilde{\theta}_k^n(t)$ exists and is finite *a.s.* for each fixed $t \geq 0$. It also (along with Remark 4.1) gives us (21).

Now to define the sets D_k on the statement of the result, for $k = 1$ take D_k the rational numbers of $[0, \infty)$. For $k > 1$, take a countable dense subset of $\{t \geq 0 : \mathbb{P}(X_{k-1}(t) \in \mathbb{N}_*^{k-1}) = 1\}$. This set is dense in $[0, \infty)$, since it has total Lebesgue measure (Remark 4.3).

Let us take these limits $\theta_k^\infty(t) := \lim_{n \rightarrow \infty} \tilde{\theta}_k^n(t)$ for $t \in D_k$. Since each $\tilde{\theta}_k^n$ is monotonic, then the limit will be monotonic as well.

With this we can define $\theta_k^\infty(t) = \lim_{s \rightarrow t+} \theta_k^\infty(s)$ for any $t \notin D_k$, this limit being taken over $s \in D_k$.

To complete the proof we only need to show that θ_k^∞ is right continuous over D_k . For a $t \in D_k$, let $\theta_k^\infty(t+) = \lim_{s \rightarrow t+} \theta_k^\infty(s)$, this limit exists almost surely because of monotonicity.

We can compute the Laplace transform of $\theta_1^\infty(t+) - \theta_1^\infty(t)$ using Proposition 4.2 and the fact that θ_1^n has stationary increments:

$$\begin{aligned} & \mathbb{E} [\exp \{ -\lambda (\theta_1^\infty(t+) - \theta_1^\infty(t)) \}] \\ &= \lim_{s \rightarrow 0+} \lim_{n \rightarrow \infty} \mathbb{E} \left[\exp \left\{ -s \sum_{x_1} h_{x|1} \left(\sum_{x_2} h_{x|2} \left(\cdots \sum_{x_n} h_{x|n} (\lambda W(x|n)) \cdots \right) \right) \right\} \right] \\ &= \lim_{s \rightarrow 0+} \mathbb{E} [\exp \{ -s \phi(\lambda) \}] = 1. \end{aligned}$$

The exchanges between limit and expected values taken here can be justified either by the continuity theorem for Laplace transforms or the dominated convergence theorem. Moreover this random function $\phi(\lambda)$ is finite *a.s.* because $\theta_1^n(t)$ is finite *a.s.*

So we have proved that $\theta_1^\infty(t+) = \theta_1^\infty(t)$ *a.s.* for every rational t . This concludes the proof for the case $k = 1$.

For the case $k > 1$, note that our choice for D_k guarantees that, with probability one, $X_{k-1}(t) \in \mathbb{N}_*^{k-1}$ for all $t \in D_k$.

Therefore t belongs to an interval $[a, b)$ such that $X_{k-1}(s) = x|_{k-1}$ for all $s \in [a, b)$. So $L_{k-1}(y|_{k-1}, s)$ is constant in this interval for every $y|_{k-1} \neq x|_{k-1}$. Using Remark 4.1

e the previous case, we can conclude:

$$\begin{aligned}
\theta_k^\infty(t+) - \theta_k^\infty(t) &= \lim_{s \rightarrow 0+} \lim_{n \rightarrow \infty} \theta_k^n(t+s) - \theta_k^n(t) \\
&= \lim_{s \rightarrow 0+} \lim_{n \rightarrow \infty} \sum_{y|_{k-1}} \left[\theta_{y|_{k-1}}^n(L_{k-1}(y|_{k-1}, t+s)) - \theta_{y|_{k-1}}^n(L_{k-1}(y|_{k-1}, t)) \right] \\
&= \lim_{s \rightarrow 0+} \lim_{n \rightarrow \infty} \theta_{x|_{k-1}}^n(L_{k-1}(x|_{k-1}, t+s)) - \theta_{x|_{k-1}}^n(L_{k-1}(x|_{k-1}, t)) \\
&= \lim_{s \rightarrow 0+} \theta_{x|_{k-1}}^\infty(L_{k-1}(x|_{k-1}, t+s)) - \theta_{x|_{k-1}}^\infty(L_{k-1}(x|_{k-1}, t)) = 0. \quad \square
\end{aligned}$$

Theorem 4.5 (Non Triviality). *Suppose that:*

$$\sum_{k=1}^{\infty} \frac{1 - \alpha_{k+1}}{1 - \alpha_k} < \infty. \quad (22)$$

Then, for every $k \in \mathbb{N}_$, θ_k^∞ is a.s. a strictly increasing function and $\lim_{t \rightarrow \infty} \theta_k^\infty(t) = \infty$.*

Theorem 4.6 (Triviality). *Suppose that:*

$$\sum_{k=1}^{\infty} (1 - \alpha_k) < \infty, \quad \sum_{k=1}^{\infty} \frac{1 - \alpha_{k+1}}{1 - \alpha_k} = \infty, \quad (23)$$

then $\theta_k^\infty(t) = 0$ a.s. for every $t \geq 0$ and $k \in \mathbb{N}_$.*

Remark 4.7. *Note that the case $\sum_k (1 - \alpha_k) = \infty$ is not covered by either Theorem 4.5 or 4.6. We believe that the limit clocks are also trivial in this case (based on a few instances where we can see computations through, and on the intuition developed on these analyses; the general case seems too hard to compensate the expected dead end result).*

We will refer to condition (22) as the non-triviality condition. Note that it implies that $\sum_i (1 - \alpha_i) < \infty$.

Both statements will be proven by studying the behavior of the random variables in the exponent of the right hand side of (19). To make notations more compact, let us define, for $n \geq k$ and a fixed $\lambda \geq 0$:

$$Z_{x|_k}^n := \begin{cases} \lambda W(x|_k), & \text{if } k = n; \\ \sum_{x_{k+1}} h_{x|_{k+1}}(Z_{x|_{k+1}}^n), & \text{otherwise.} \end{cases} \quad (24)$$

Let us prove an auxiliary result that will be useful in the proof of both theorems:

Lemma 4.8. Suppose that $\sum_i 1 - \alpha_i < \infty$. Then $\sum_i \frac{1 - \alpha_{i+1}}{1 - \alpha_i} < \infty$ if and only if $\sum_i (1 - d_i) < \infty$, where:

$$d_i := \frac{\alpha_i \Gamma(\alpha_i) \Gamma(1 - \alpha_i)}{\Gamma(1 - \alpha_i / \alpha_{i+1})}.$$

Proof. Using that $\Gamma(1 + \alpha) = \alpha \Gamma(\alpha)$ and developing the definition of d_i , we can write:

$$\begin{aligned} & \Gamma\left(2 - \frac{\alpha_i}{\alpha_{i+1}}\right) \frac{(1 - d_i)(1 - \alpha_i)}{1 - \alpha_{i+1}} \\ &= \frac{1 - \alpha_i}{1 - \alpha_{i+1}} \left[\Gamma\left(2 - \frac{\alpha_i}{\alpha_{i+1}}\right) - \Gamma(2 - \alpha_i) \right] \end{aligned} \quad (25)$$

$$+ \frac{\Gamma(2 - \alpha_i)}{1 - \alpha_{i+1}} \left[1 - \alpha_i - \left(1 - \frac{\alpha_i}{\alpha_{i+1}}\right) \right] \quad (26)$$

$$+ \frac{\Gamma(2 - \alpha_i)}{1 - \alpha_{i+1}} \left(1 - \frac{\alpha_i}{\alpha_{i+1}}\right) [1 - \Gamma(1 + \alpha_i)]. \quad (27)$$

Note that the three terms are positive for large enough i , since Γ is decreasing near 1 and increasing near 2 and $0 < \alpha_i < \alpha_i / \alpha_{i+1} < 1$.

Since Γ is differentiable in $[1, 2]$, we have that (25) converges to zero as $i \rightarrow \infty$. A straightforward computation shows that (26) converges to 1 as $i \rightarrow \infty$.

Let $a_i := (25) + (26)$ and $b_i := (27)$, we can write:

$$b_i \frac{1 - \alpha_{i+1}}{1 - \alpha_i} = \frac{\alpha_{i+1} - \alpha_i}{\alpha_{i+1}} \frac{\Gamma(2) - \Gamma(1 + \alpha_i)}{1 - \alpha_i}.$$

Therefore $b'_i := b_i \frac{1 - \alpha_{i+1}}{1 - \alpha_i}$ is summable, since $\sum_i (1 - \alpha_i) < \infty$ and $\frac{\Gamma(2) - \Gamma(1 + \alpha_i)}{1 - \alpha_i}$ converges to a constant as $i \rightarrow \infty$.

Finally, we can write:

$$(1 - d_i) \Gamma\left(2 - \frac{\alpha_i}{\alpha_{i+1}}\right) = a_i \frac{1 - \alpha_{i+1}}{1 - \alpha_i} + b'_i.$$

Since $a_i \rightarrow 1$ and $\Gamma(2 - \alpha_i / \alpha_{i+1}) \rightarrow 1$ as $i \rightarrow \infty$ and b'_i is summable, we conclude that $\sum_i (1 - d_i) < \infty$ if and only if $\sum_i \frac{1 - \alpha_{i+1}}{1 - \alpha_i} < \infty$. \square

Proof of Theorem 4.6. We will only prove that $\theta_1^\infty \equiv 0$. It is straightforward to extend this case to the general case.

Because of Proposition 4.2 and (24), it is enough to prove that Z_\emptyset^n converges to 0 in probability as n goes to infinity. We will show the stronger result that $\mathbb{E}(Z_\emptyset^n) \rightarrow 0$ as $n \rightarrow \infty$.

Let us define:

$$a_k^n = \begin{cases} \mathbb{E}(Z_{x|k}^n) & \text{if } k < n, \\ \mathbb{E}\left(\left(Z_{x|k}^n\right)^{\alpha_k}\right) & \text{if } k = n. \end{cases}$$

Using the last claim in Proposition 3.4, together with Proposition 3.2, we can compute $a_n^n = \lambda^{\alpha_n} \Gamma(1 - \alpha_n/\alpha_{n+1})/\Gamma(1 - \alpha_n)$. Using Proposition 3.1, together with Campbell's Theorem and Jensen's Inequality, we can write that for $k < n$:

$$\begin{aligned} a_{k-1}^n &= \mathbb{E} \left[\sum_{x_k} \frac{\gamma_k(x|k) Z_{x|k}^n}{1 + \gamma_k(x|k) Z_{x|k}^n} \right] \\ &= \int_0^\infty \frac{x}{1+x} \frac{\mathbb{E} \left[(Z_{x|k}^n)^{\alpha_k} \right] c_k}{x^{1+\alpha_k}} dx \\ &\leq c_k (a_k^n)^{\alpha_k} \int_0^\infty \frac{1}{x^{\alpha_k} (1+x)} dx \\ &= c_k (a_k^n)^{\alpha_k} \int_0^1 y^{1-\alpha_k-1} (1-y)^{\alpha_k-1} dy \\ &= c_k (a_k^n)^{\alpha_k} \Gamma(1 - \alpha_k) \Gamma(\alpha_k) \\ &= (a_k^n)^{\alpha_k} \frac{\alpha_k \Gamma(\alpha_k) \Gamma(1 - \alpha_k)}{\Gamma\left(1 - \frac{\alpha_k}{\alpha_{k+1}}\right)}. \end{aligned} \tag{28}$$

In the last line we substituted the value of c_k in (1). We can compute a_{n-1}^n in an analogous way, but using the actual value of $\mathbb{E}[(Z_{x|n}^n)^{\alpha_n}]$ instead of estimating it via Jensen's inequality, obtaining:

$$a_{n-1}^n = a_n^n \frac{\alpha_n \Gamma(\alpha_n) \Gamma(1 - \alpha_n)}{\Gamma\left(1 - \frac{\alpha_n}{\alpha_{n+1}}\right)} = \lambda^{\alpha_n} \alpha_n \Gamma(\alpha_n) \tag{29}$$

Iterating on (28) and using this equality, we obtain that:

$$\begin{aligned} a_0^n &\leq \prod_{i=1}^{n-1} \left[\frac{\alpha_i \Gamma(\alpha_i) \Gamma(1 - \alpha_i)}{\Gamma\left(1 - \frac{\alpha_i}{\alpha_{i+1}}\right)} \right]^{\alpha_1 \dots \alpha_{i-1}} (a_{n-1}^n)^{\alpha_1 \dots \alpha_{n-1}} \\ &= (\lambda^{\alpha_n} \alpha_n \Gamma(\alpha_n))^{\alpha_1 \dots \alpha_{n-1}} \prod_{i=1}^{n-1} \left[\frac{\alpha_i \Gamma(\alpha_i) \Gamma(1 - \alpha_i)}{\Gamma\left(1 - \frac{\alpha_i}{\alpha_{i+1}}\right)} \right]^{\alpha_1 \dots \alpha_{i-1}} \end{aligned} \tag{30}$$

Note that $\lambda^{\prod_j \alpha_j} \leq \max\{\lambda, 1\}$ and that $\alpha_n \Gamma(\alpha_n) \rightarrow 1$ as $n \rightarrow \infty$. So if we show that the product on (30) converges to zero as $n \rightarrow \infty$, it will follow that $a_0^n \xrightarrow{n \rightarrow \infty} 0$. This motivates the definition:

$$d_i := \frac{\alpha_i \Gamma(\alpha_i) \Gamma(1 - \alpha_i)}{\Gamma\left(1 - \frac{\alpha_i}{\alpha_{i+1}}\right)}, \quad b_i := d_i^{\alpha_1 \dots \alpha_{i-1}}. \quad (31)$$

Now we want to show that $\prod_{i=1}^{\infty} b_i = 0$. Note that $d_i, b_i \in (0, 1)$, since Γ is a decreasing function on $(0, 1)$ and $\alpha \Gamma(\alpha) = \Gamma(\alpha + 1) < 1$ for $\alpha \in (0, 1)$.

By assumption, $\sum_i (1 - \alpha_i) < \infty$. This implies that $\prod_i \alpha_i > 0$. Then:

$$\begin{aligned} \prod_i b_i = 0 &\Leftrightarrow \sum_i \log b_i = -\infty \Leftrightarrow \sum_i \alpha_1 \dots \alpha_{i-1} \log d_i = -\infty \\ &\Leftrightarrow \sum_i \log d_i = -\infty \Leftrightarrow \prod_i d_i = 0 \Leftrightarrow \sum_i (1 - d_i) = +\infty. \end{aligned}$$

Finally Lemma 4.8 guarantees that $\sum_i (1 - d_i) = \infty$ whenever $\sum_i \frac{1 - \alpha_{i+1}}{1 - \alpha_i} = \infty$. \square

Before proving Theorem 4.5, let us state an auxiliary result:

Lemma 4.9. *Let $\{\gamma_i : i \in \mathbb{N}_*\}$ be the marks of a Poisson Process with intensity measure $\mu(dx) = \frac{c}{x^{1+\alpha}} \mathbb{I}\{x > 0\}$, for some $c > 0$ and $\alpha \in (0, 1)$. Taking $X := \sum_i \frac{\gamma_i}{1 + \gamma_i}$ and fixing an $\beta \in (0, 1)$, it is true that:*

$$\mathbb{E}(X^\beta) \geq \frac{c \Gamma(\alpha) \Gamma(1 - \alpha)}{[1 + c \Gamma(\alpha) \Gamma(1 - \alpha)]^{1-\beta}}.$$

Proof. Take $\phi(\theta) := \mathbb{E}(e^{-\theta X})$ the Laplace transform of X . We compute this quantity using Campbell's Theorem:

$$\begin{aligned} \phi(\theta) &= \exp \left\{ - \int_0^\infty (1 - e^{-\theta \frac{x}{x+1}}) \frac{c}{x^{1+\alpha}} dx \right\} \\ &= \exp \left\{ -c \int_0^1 \frac{1 - e^{-\theta y}}{y^{1+\alpha} (1 - y)^{1-\alpha}} dy \right\} \\ &=: \exp \{ -c \psi(\theta) \}. \end{aligned}$$

The first derivative of ψ can be computed and estimated as:

$$\begin{aligned}
\psi'(\theta) &= \lim_{h \rightarrow 0} \frac{\psi(\theta + h) - \psi(\theta)}{h} \\
&= \lim_{h \rightarrow 0} \int_0^1 \frac{e^{-\theta y}}{y^{1+\alpha}(1-y)^{1-\alpha}} \frac{1 - e^{-hy}}{h} dy \\
&= \int_0^1 \frac{e^{-\theta y}}{y^\alpha(1-y)^{1-\alpha}} dy \\
&\geq \int_0^1 \frac{e^{-\theta}}{y^\alpha(1-y)^{1-\alpha}} dy \\
&= e^{-\theta} \Gamma(\alpha) \Gamma(1-\alpha),
\end{aligned}$$

justifying the equality between the second and third line by the Dominated Convergence Theorem.

Since $1 - e^{-x} \leq x$, we can write:

$$\begin{aligned}
\psi(\theta) &= \int_0^1 \frac{1 - e^{-\theta y}}{y^{1+\alpha}(1-y)^{1-\alpha}} dy \\
&\leq \theta \int_0^1 \frac{1}{y^\alpha(1-y)^{1-\alpha}} dy \\
&= \theta \Gamma(\alpha) \Gamma(1-\alpha) \\
\phi(\theta) &= e^{-c\psi(\theta)} \\
&\geq e^{-c\theta \Gamma(\alpha) \Gamma(1-\alpha)}.
\end{aligned}$$

Finally we can conclude:

$$\begin{aligned}
\mathbb{E}(X^\beta) &= \mathbb{E}(X^{1-(1-\beta)}) = -\frac{1}{\Gamma(1-\beta)} \int_0^\infty \theta^{(1-\beta)-1} \phi'(\theta) d\theta \\
&= \frac{1}{\Gamma(1-\beta)} \int_0^\infty \theta^{-\beta} c\phi(\theta) \psi'(\theta) d\theta \\
&\geq \frac{c}{\Gamma(1-\beta)} \int_0^\infty \theta^{-\beta} e^{-c\theta \Gamma(\alpha) \Gamma(1-\alpha)} e^{-\theta} \Gamma(\alpha) \Gamma(1-\alpha) d\theta \\
&= \frac{c\Gamma(\alpha) \Gamma(1-\alpha)}{[1 + c\Gamma(\alpha) \Gamma(1-\alpha)]^{1-\beta}}.
\end{aligned}$$

□

Proof of Theorem 4.5. Again we will show the claim only to θ_1^∞ . We can extend the proof to the general case in a straightforward manner by using the fact that $\tilde{\theta}_1^n = \tilde{\theta}_{k+1}^n \circ \theta_1^k$.

Let us define $a_k^n := \mathbb{E} \left[(Z_{x|_k}^n)^{\alpha_k} \right]$. We will show that $\liminf_{n \rightarrow \infty} a_1^n > 0$. Using Propositions 4.2 and 3.1 we can write:

$$\mathbb{E} \left[e^{-\lambda \tilde{\theta}_1^n(t)} \right] = \mathbb{E} \left[\exp \left\{ -t \sum_{x1} \gamma_1(x|_1) Z_{x|_1}^n \right\} \right] = \mathbb{E} \left[\exp \left\{ -t \sum_{x1} \gamma_1(x|_1) (a_1^n)^{1/\alpha_1} \right\} \right].$$

If we show that $\liminf_{n \rightarrow \infty} a_1^n > 0$, it will follow from the expression above that $\lim_{t \rightarrow \infty} \theta_1^\infty(t) = \infty$ in probability. Since θ_1^∞ is *a.s.* non decreasing, it follows that $\lim_{t \rightarrow \infty} \theta_1^\infty(t) = \infty$ *a.s.*

To show that $\liminf_{n \rightarrow \infty} a_1^n > 0$, let us start by applying Lemma 4.9 and Proposition 3.1 to write that for $k \leq n$:

$$\begin{aligned} a_{k-1}^n &\geq \frac{c_k a_k^n \Gamma(\alpha_k) \Gamma(1 - \alpha_k)}{[1 + c_k a_k^n \Gamma(\alpha_k) \Gamma(1 - \alpha_k)]^{1 - \alpha_{k-1}}} \\ &= \frac{a_k^n \frac{\alpha_k \Gamma(\alpha_k) \Gamma(1 - \alpha_k)}{\Gamma(1 - \alpha_k / \alpha_{k+1})}}{\left[1 + a_k^n \frac{\alpha_k \Gamma(\alpha_k) \Gamma(1 - \alpha_k)}{\Gamma(1 - \alpha_k / \alpha_{k+1})} \right]^{1 - \alpha_{k-1}}} \end{aligned} \quad (32)$$

Let us first work with the denominator. For this look at (30). Although the definition of a_k^n is slightly different on that proof, we can still use Jensen's inequality to obtain:

$$\begin{aligned} a_k^n &:= \mathbb{E} \left[(Z_{x|_k}^n)^{\alpha_k} \right] \leq (\mathbb{E} [Z_{x|_k}^n])^{\alpha_k} \\ &\leq \left((\lambda^{\alpha_n} \alpha_n \Gamma(\alpha_n))^{\alpha_{k+1} \dots \alpha_{n-1}} \prod_{i=k+1}^{n-1} \left[\frac{\alpha_i \Gamma(\alpha_i) \Gamma(1 - \alpha_i)}{\Gamma \left(1 - \frac{\alpha_i}{\alpha_{i+1}} \right)} \right]^{\alpha_{k+1} \dots \alpha_{i-1}} \right)^{\alpha_k} \\ &\leq \lambda^{\alpha_k \dots \alpha_n} \leq \max\{\lambda, 1\} \end{aligned}$$

Letting $\delta := \max\{\lambda, 1\}$ and using this expression on the denominator of (32), we obtain:

$$a_{k-1}^n \geq a_k^n (1 + \delta)^{-(1 - \alpha_{k-1})} \frac{\alpha_k \Gamma(\alpha_k) \Gamma(1 - \alpha_k)}{\Gamma(1 - \alpha_k / \alpha_{k+1})}. \quad (33)$$

Knowing that $0 < \alpha_i < \alpha_i / \alpha_{i+1} < 1$ and Γ is a decreasing function near zero, we can iterate this inequality to obtain:

$$\begin{aligned} a_k^n &\geq \lambda^{\alpha_n} \frac{\Gamma \left(1 - \frac{\alpha_n}{\alpha_{n+1}} \right)}{\Gamma(1 - \alpha_n)} (1 + \delta)^{-\sum_{j=k}^{n-1} (1 - \alpha_j)} \prod_{j=k+1}^n \frac{\alpha_j \Gamma(\alpha_j) \Gamma(1 - \alpha_j)}{\Gamma \left(1 - \frac{\alpha_j}{\alpha_{j+1}} \right)} \\ &\geq \lambda^{\alpha_n} (1 + \delta)^{-\sum_{j=k}^{n-1} (1 - \alpha_j)} \prod_{j=k+1}^n \frac{\alpha_j \Gamma(\alpha_j) \Gamma(1 - \alpha_j)}{\Gamma \left(1 - \frac{\alpha_j}{\alpha_{j+1}} \right)}. \end{aligned} \quad (34)$$

Note that the terms in the product at the end of this last expression are exactly equal to d_i , defined in Lemma 4.8. We have shown in that Lemma that, whenever $\sum_i \frac{1-\alpha_{i+1}}{1-\alpha_i} < \infty$, we have that $\sum_i (1-d_i) < \infty$, which implies that $\prod_i d_i > 0$.

Therefore we conclude that $\liminf_{n \rightarrow \infty} a_k^n > 0$. To complete the proof of the theorem, we have to show that θ_1^∞ is *a.s.* strictly increasing. We will do this by showing that $\lim_{\lambda \rightarrow \infty} \liminf_{n \rightarrow \infty} Z_\emptyset^n(\lambda) = \infty$ in probability.

Since we are interested now only on large λ , we can assume $\lambda > 1$. Taking a constant $C > 0$ such that $\prod_{i=1}^\infty d_i > C$ and using the definition of δ , we can rewrite (34) as:

$$\begin{aligned} a_k^n(\lambda) &\geq C\lambda^{\alpha_n}(1+\lambda)^{-\sum_{j=k}^{n-1}(1-\alpha_j)}, \\ \liminf_{n \rightarrow \infty} a_k^n(\lambda) &\geq C\lambda(1+\lambda)^{-\sum_{j=k}^\infty(1-\alpha_j)}. \end{aligned}$$

Now fix a k such that $\sum_{i \geq k} (1-\alpha_i) < 1$. From the inequality above, we may conclude that $\lim_{\lambda \rightarrow \infty} \liminf_{n \rightarrow \infty} a_k^n(\lambda) = \infty$.

Using Proposition 3.1, and taking an arbitrary $M \in \mathbb{N}$, we can write:

$$\begin{aligned} Z_{x|_{k-1}}^n &= \sum_{x_k} \frac{\gamma_k(x|_k) Z_{x|_k}^n(\lambda)}{1 + \gamma_k(x|_k) Z_{x|_k}^n(\lambda)} \\ &\stackrel{\mathcal{D}}{=} \sum_{x_k} \frac{\gamma_k(x|_k) (a_k^n(\lambda))^{1/\alpha_k}}{1 + \gamma_k(x|_k) (a_k^n(\lambda))^{1/\alpha_k}} \\ &\geq \sum_{x_k=1}^M \frac{\gamma_k(x|_k) (a_k^n(\lambda))^{1/\alpha_k}}{1 + \gamma_k(x|_k) (a_k^n(\lambda))^{1/\alpha_k}} \xrightarrow[n \rightarrow \infty, \lambda \rightarrow \infty]{a.s.} M. \end{aligned}$$

Since M is arbitrary, it follows that $\lim_{\lambda \rightarrow \infty} \liminf_{n \rightarrow \infty} Z_{x|_{k-1}}^n(\lambda) = \infty$ in probability. Knowing that $Z_{x|_{k-2}}^n = \sum_{x_{k-1}} h_{x|_{k-1}}(Z_{x|_{k-1}}^n)$, we can use analogous arguments to show that $\lim_{\lambda \rightarrow \infty} \liminf_{n \rightarrow \infty} Z_{x|_{k-2}}^n = \infty$ in probability. Iterating we conclude that:

$$\lim_{\lambda \rightarrow \infty} \liminf_{n \rightarrow \infty} Z_\emptyset^n(\lambda) = \infty \text{ in probability.} \quad \square$$

5 Convergence

A natural question to ask after constructing the K process on a tree with infinite depth (\mathbb{Y}) is whether this process is the limit, in some sense, of the K processes on trees with finite depth (X_k) as the depth grows to infinity. We will address this question in this section and prove Theorem 2.7 stated in Section 2.

Lemma 5.1. *Suppose that $\frac{1-\alpha_{k+1}}{1-\alpha_k} \rightarrow 0$ as $k \rightarrow \infty$. Then:*

$$\lim_{k \rightarrow \infty} \mathbb{E} [|W(x|_k) - 1|^{\alpha_k}] = 0 \quad (35)$$

We note that the condition of Lemma 5.1 holds under the non-triviality condition (22).

Proof. We know the Laplace transform of $W(x|_k)$ from Proposition 3.4. With it we can apply Proposition 1.1.12 from [13] to obtain the characteristic function of $W(x|_k) - 1$:

$$\begin{aligned} \varphi_k(u) &:= \mathbb{E} [e^{iu(W(x|_k)-1)}] \\ &= \exp \left\{ -|u|^{\alpha_{k+1}} \left[\cos \left(\frac{\pi \alpha_{k+1}}{2} \right) - i \operatorname{sgn}(u) \sin \left(\frac{\pi \alpha_{k+1}}{2} \right) \right] - iu \right\} \\ &= \exp \left\{ -|u|^{\alpha_{k+1}} \cos \left(\frac{\pi \alpha_{k+1}}{2} \right) - i \left[u - |u|^{\alpha_{k+1}} \operatorname{sgn}(u) \sin \left(\frac{\pi \alpha_{k+1}}{2} \right) \right] \right\}. \end{aligned} \quad (36)$$

Theorem 2.2 from [14] states that:

$$\mathbb{E} [|W(x|_k) - 1|^{\alpha_k}] = \frac{1}{\cos \left(\frac{\pi \alpha_k}{2} \right)} \operatorname{Re} \left[\frac{\alpha_k}{\Gamma(1 - \alpha_k)} \int_0^\infty \frac{1 - \varphi_k(-u)}{u^{1+\alpha_k}} du \right].$$

Note that $\frac{\alpha_k}{\cos \left(\frac{\pi \alpha_k}{2} \right) \Gamma(1 - \alpha_k)}$ converges to $\frac{2}{\pi}$ as $k \rightarrow \infty$. So we are left with showing that the real part of the integral converges to zero. Fixing an arbitrary $\epsilon > 0$ we can write:

$$\begin{aligned} &\operatorname{Re} \left[\int_0^\infty \frac{1 - \varphi_k(-u)}{u^{1+\alpha_k}} du \right] \\ &= \int_0^\infty \frac{1}{u^{1+\alpha_k}} \left[1 - \exp \left\{ -u^{\alpha_{k+1}} \cos \left(\frac{\pi \alpha_{k+1}}{2} \right) \right\} \cos \left(u - u^{\alpha_{k+1}} \sin \left(\frac{\pi \alpha_{k+1}}{2} \right) \right) \right] du \\ &= \int_0^\epsilon \frac{1}{u^{1+\alpha_k}} \left[1 - \exp \left\{ -u^{\alpha_{k+1}} \cos \left(\frac{\pi \alpha_{k+1}}{2} \right) \right\} \cos \left(u - u^{\alpha_{k+1}} \sin \left(\frac{\pi \alpha_{k+1}}{2} \right) \right) \right] du \end{aligned} \quad (37)$$

$$+ \int_\epsilon^\infty \frac{1}{u^{1+\alpha_k}} \left[1 - \exp \left\{ -u^{\alpha_{k+1}} \cos \left(\frac{\pi \alpha_{k+1}}{2} \right) \right\} \cos \left(u - u^{\alpha_{k+1}} \sin \left(\frac{\pi \alpha_{k+1}}{2} \right) \right) \right] du. \quad (38)$$

To control (38), note that the integrand converges to zero as $k \rightarrow \infty$ and can be bounded by $2/u^{1+\alpha_k} \leq 2/u^{3/2}$ for big enough k . Therefore, by dominated convergence, (38) converges to zero as $k \rightarrow \infty$ for any choice of $\epsilon > 0$.

We control (37), using $1 - e^{-x} \leq x$, as follows. It is bounded above by

$$\begin{aligned}
& \int_0^\epsilon \frac{1}{u^{1+\alpha_k}} \left[u^{\alpha_{k+1}} \cos\left(\frac{\pi\alpha_{k+1}}{2}\right) - \log \cos\left(u - u^{\alpha_{k+1}} \sin\left(\frac{\pi\alpha_{k+1}}{2}\right)\right) \right] du \\
&= \int_0^\epsilon \frac{1}{u^{1+\alpha_k}} u^{\alpha_{k+1}} \cos\left(\frac{\pi\alpha_{k+1}}{2}\right) du - \int_0^\epsilon \frac{1}{u^{1+\alpha_k}} \log \cos\left(u - u^{\alpha_{k+1}} \sin\left(\frac{\pi\alpha_{k+1}}{2}\right)\right) du \\
&= \frac{\epsilon^{\alpha_{k+1}-\alpha_k}}{\alpha_{k+1}-\alpha_k} \cos\left(\frac{\pi\alpha_{k+1}}{2}\right) - \int_0^\epsilon \frac{1}{u^{1+\alpha_k}} \log \cos\left(u - u^{\alpha_{k+1}} \sin\left(\frac{\pi\alpha_{k+1}}{2}\right)\right) du. \quad (39)
\end{aligned}$$

Since $\frac{1-\alpha_{k+1}}{1-\alpha_k} \xrightarrow{k \rightarrow \infty} 0$ by hypothesis and $\cos(\pi x/2)/(1-x) \xrightarrow{x \rightarrow 1} \frac{\pi}{2}$, we can rewrite the leftmost term from (39) as:

$$\begin{aligned}
\frac{\epsilon^{\alpha_{k+1}-\alpha_k}}{\alpha_{k+1}-\alpha_k} \cos\left(\frac{\pi\alpha_{k+1}}{2}\right) &= \epsilon^{\alpha_{k+1}-\alpha_k} \frac{1-\alpha_{k+1}}{\alpha_{k+1}-\alpha_k} \frac{\cos\left(\frac{\pi\alpha_{k+1}}{2}\right)}{1-\alpha_{k+1}} \\
&= \epsilon^{\alpha_{k+1}-\alpha_k} \left(\frac{1-\alpha_k}{1-\alpha_{k+1}} - 1 \right)^{-1} \frac{\cos\left(\frac{\pi\alpha_{k+1}}{2}\right)}{1-\alpha_{k+1}} \xrightarrow{k \rightarrow \infty} 0.
\end{aligned}$$

To control the integral in (39), let us first remark that there exists an $\epsilon_0 > 0$ such that if $|x| < \epsilon_0$ then $-\log \cos x < x^2$. With this, for small enough $\epsilon > 0$ we can write:

$$\begin{aligned}
& - \int_0^\epsilon \frac{1}{u^{1+\alpha_k}} \log \cos\left(u - u^{\alpha_{k+1}} \sin\left(\frac{\pi\alpha_{k+1}}{2}\right)\right) du \\
& \leq \int_0^\epsilon \frac{1}{u^{1+\alpha_k}} \left(u - u^{\alpha_{k+1}} \sin\left(\frac{\pi\alpha_{k+1}}{2}\right) \right)^2 du \\
&= \int_0^\epsilon u^{1-\alpha_k} - 2u^{\alpha_{k+1}-\alpha_k} \sin\left(\frac{\pi\alpha_{k+1}}{2}\right) + u^{2\alpha_{k+1}-\alpha_k-1} \sin^2\left(\frac{\pi\alpha_{k+1}}{2}\right) du \\
&= \frac{\epsilon^{2-\alpha_k}}{2-\alpha_k} - 2 \frac{\epsilon^{\alpha_{k+1}-\alpha_k+1}}{\alpha_{k+1}-\alpha_k+1} \sin\left(\frac{\pi\alpha_{k+1}}{2}\right) + \frac{\epsilon^{2\alpha_{k+1}-\alpha_k}}{2\alpha_{k+1}-\alpha_k} \sin^2\left(\frac{\pi\alpha_{k+1}}{2}\right) \\
& \xrightarrow{k \rightarrow \infty} 0. \quad \square
\end{aligned}$$

Proposition 5.2 (Finite dimensional convergence). *If $\sum_k (1-\alpha_k) < \infty$, then for every fixed $t > 0$ and $k \in \mathbb{N}_*$, $\theta_k^n(t)$ converges in probability to $\theta_k^\infty(t)$ as $n \rightarrow \infty$.*

Proof. We will assume that $\frac{1-\alpha_{k+1}}{1-\alpha_k} \rightarrow 0$ as $k \rightarrow \infty$. When this is not true then the conditions of Theorem 4.6 hold and we can use an analogous argument to show that $\theta_k^n(t) \xrightarrow{n \rightarrow \infty} 0$ in probability for every $t > 0$.

Let us first show the case $k = 1$. Taking $Z_{x|n} = |W(x|n) - 1|$ in the proof of

Proposition 4.2 we obtain:

$$\begin{aligned} \mathbb{E} \left[\exp \left\{ -|\theta_1^n(t) - \tilde{\theta}_1^n(t)| \right\} \right] &\geq \mathbb{E} \left[\exp \left\{ -\sum_{x|_n} L(x|_n, \theta_1^n(t)) |1 - W(x|_n)| \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ -t \sum_{x_1} h_{x|_1} \left(\sum_{x_2} h_{x|_2} \left(\cdots \sum_{x_n} h_{x|_n} (|W(x|_n) - 1|) \cdots \right) \right) \right\} \right]. \end{aligned} \quad (40)$$

Now proceeding as in Theorem 4.6, let us define:

$$Z_{x|_k}^n := \begin{cases} \sum_{x_{k+1}} h_{x|_{k+1}} (Z_{x|_{k+1}}^n), & \text{if } k < n; \\ |W(x|_n) - 1|, & \text{if } k = n \end{cases}; \quad a_k^n := \begin{cases} \mathbb{E} [Z_{x|_k}^n], & \text{if } k < n; \\ \mathbb{E} [|W(x|_n) - 1|^{\alpha_n}], & \text{if } k = n. \end{cases}$$

Lemma 5.1 states that $a_n^n \rightarrow 0$ as $n \rightarrow \infty$. Following the proof of Theorem 4.6, we get (28) and the first equality of (29), from which we obtain that $a_{k-1}^n \leq a_k^n$ for every $k \leq n$. Therefore $a_k^n \leq a_n^n \xrightarrow{n \rightarrow \infty} 0$ for every $k < n$ and $Z_{\emptyset}^n \rightarrow 0$ in the L_1 norm as $n \rightarrow \infty$. By applying the Dominated Convergence Theorem on (40), we conclude the case $k = 1$.

For the general case, using Remark 4.1 and the previous result, we can write:

$$\begin{aligned} \mathbb{E} \left[e^{-|\theta_{k+1}^n(t) - \tilde{\theta}_{k+1}^n(t)|} \right] &\geq \mathbb{E} \left[\exp \left\{ -\sum_{x|_k} \left| \theta_{x|_k}^n(L(x|_k, t)) - \tilde{\theta}_{x|_k}^n(L(x|_k, t)) \right| \right\} \right] \\ &= \mathbb{E} \left[\exp \left\{ -\sum_{x|_k} L(x|_k, t) Z_{x|_k}^n \right\} \right]. \end{aligned} \quad (41)$$

Note that, since $L(x|_k, t)$ is independent from $Z_{x|_k}^n$, then:

$$\mathbb{E} \left[\sum_{x|_k} L(x|_k, t) Z_{x|_k}^n \right] = \sum_{x|_k} \mathbb{E} [L(x|_k, t) Z_{x|_k}^n] = t a_k^n \xrightarrow{n \rightarrow \infty} 0.$$

We conclude the proof by applying the Dominated Convergence Theorem on (41). \square

Lemma 5.3. *Almost surely:*

$$\sup_n \sum_{x|_n} \bar{\gamma}_n(x|_n) < \infty. \quad (42)$$

Proof. Let $A_n := \{x|_n \in \mathbb{N}_*^n : \bar{\gamma}_n(x|_n) > 1\}$ and $m_n := \max\{\bar{\gamma}_n(x|_n) : x|_n \in \mathbb{N}_*^n\}$. Using Proposition 3.4, we know that almost surely:

$$\begin{aligned} W(\emptyset) &= \lim_{n \rightarrow \infty} \sum_{x|_n \in \mathbb{N}_*^n} (\bar{\gamma}_n(x|_n))^{\alpha_{n+1}} \geq \limsup_{n \rightarrow \infty} \sum_{x|_n \in A_n} (\bar{\gamma}_n(x|_n))^{\alpha_{n+1}} \geq \limsup_{n \rightarrow \infty} |A_n|; \\ W(\emptyset) &= \lim_{n \rightarrow \infty} \sum_{x|_n \in \mathbb{N}_*^n} (\bar{\gamma}_n(x|_n))^{\alpha_{n+1}} \geq \limsup_{n \rightarrow \infty} (m_n)^{\alpha_{n+1}} = \limsup_{n \rightarrow \infty} m_n. \end{aligned}$$

Since $W(\emptyset) < \infty$, then $\limsup_n |A_n| < \infty$ and $\limsup_n m_n < \infty$ *a.s.* With this we can write:

$$\begin{aligned} \sum_{x|_n} \bar{\gamma}_n(x|_n) &= \sum_{x|_n \in A_n} \bar{\gamma}_n(x|_n) + \sum_{x|_n \notin A_n} \bar{\gamma}_n(x|_n) \\ &\leq m_n |A_n| + \sum_{x|_n \notin A_n} (\bar{\gamma}_n(x|_n))^{\alpha_{n+1}}. \end{aligned}$$

Because of the previous remark, the first term of this last sum is bounded *a.s.* by a constant, while the lim sup of the second term is dominated by $W(\emptyset)$. This concludes this proof. \square

Lemma 5.4. *For almost every $\underline{\gamma}$ we have that θ_1^n is a subordinator for every $n \in \mathbb{N}_*$. Furthermore:*

$$\mathbb{E}_\gamma [\theta_1^n(t)] = t \sum_{x|_n} \bar{\gamma}_n(x|_n).$$

Proof. The expected value is computed in the proof of Lemma 4.5 from [1]. Let us prove that θ_1^n is a subordinator. We will do this by induction on n . The case $n = 1$ is a direct consequence of the independent and stationary increments of a Poisson point process and the fact the variables used to construct $\theta_1^1 = \Xi_1$ are independent.

Assuming $n \geq 2$, note that for $t, s > 0$:

$$\theta_1^n(t+s) - \theta_1^n(t) = \sum_x \sum_{i=N^{n,x}(\theta_1^{n-1}(t))+1}^{N^{n,x}(\theta_1^{n-1}(t+s))} \gamma_n(X_{n-1}(\sigma_i^{n,x})x) T_i^{n,x}.$$

Fix $0 = t_0 < t_1 < \dots < t_k$ and let us look at the joint distribution of $(\theta_1^n(t_i) - \theta_1^n(t_{i-1}))_{i=1,\dots,k}$.

$\theta_1^n(t_i) - \theta_1^n(t_{i-1})$, for varying values of i , depends on disjoint intervals of the Poisson processes $N^{n,x}$. Each one with length $\theta_1^{n-1}(t_i) - \theta_1^{n-1}(t_{i-1})$. Because of the induction hypothesis these lengths are independent and each one has the same law as $\theta_1^{n-1}(t_i - t_{i-1})$.

Note that, as can be justified by Remark 2.5, we have that for fixed $t > 0$, $X_{n-1}(\theta_1^{n-1}(t)) = (\infty, \dots, \infty)$ a.s.. The instant $\theta_1^{n-1}(t)$ is thus a renewal time for X_{n-1} , and, by construction, the law of X_{n-1} right after such a renewal is the same as the law right after the instant zero.

$\theta_1^n(t_i) - \theta_1^n(t_{i-1})$ also depends on the values of $X_{n-1}(r)$ for $r \in (\theta_1^{n-1}(t_{i-1}), \theta_1^{n-1}(t_i))$. Both ends of this interval are renewals, so what happens to X_{n-1} inside such an interval is independent of what happens on other such intervals (namely, $(\theta_1^{n-1}(t_{j-1}), \theta_1^{n-1}(t_j))$, $j \neq i$).

Therefore we have that $\theta_1^n(t_i) - \theta_1^n(t_{i-1})$, $i \geq 1$, are independent, and the law of each $\theta_1^n(t_i) - \theta_1^n(t_{i-1})$ depends only on $t_i - t_{i-1}$. That is, θ_1^n has independent and stationary increments. \square

Remark 5.5. *If the random environment $\{\gamma_k(x|_k) : k \in \mathbb{N}_*, x|_k \in \mathbb{N}_*^k\}$ is fixed, then Proposition 5.2 and Lemma 5.4 readily imply that θ_1^∞ is also a subordinator.*

Theorem 5.6. *Assuming that $\sum_k (1 - \alpha_k) < \infty$, then for every $k \in \mathbb{N}_*$, $\theta_k^n - \theta_k^\infty$ converges weakly to the identically null function in the Skorohod topology as $n \rightarrow \infty$.*

Remark 5.7. *This is a stronger result than simply stating that θ_k^n converges weakly to θ_k^∞ as $n \rightarrow \infty$ in the Skorohod topology. Convergence in the Skorohod topology to a continuous function is equivalent to uniform convergence on compact sets.*

Proof of Theorem 5.6. Under the triviality condition (23), this theorem is a direct corollary of Proposition 5.2, together with the observation that each θ_k^n is an increasing function. From now on let us assume the non-triviality condition (22).

Let us start with the case $k = 1$. Fix $\underline{\gamma}$ and let us show the convergence for almost every such choice.

Since we have shown the finite dimensional convergence in Proposition 5.2, then Theorem 7.8 from Chapter 3 of [15] says that if $\{|\theta_1^n - \theta_1^\infty|\}$ is relatively compact, then it is true that it converges weakly to the identically null function.

Using part (b) from Theorem 8.6 from Chapter 3 of [15], to show relative compactness it is enough to show that for and $0 < s < \delta$ and $t > 0$:

$$\mathbb{E}_\gamma [|(\theta_1^n(t+s) - \theta_1^\infty(t+s)) - (\theta_1^n(t) - \theta_1^\infty(t))| | \mathcal{H}_t^n] \leq 2\delta \sup_{m \in \mathbb{N}} \sum_{x|_m} \bar{\gamma}_m(x|_m) \xrightarrow[a.s.]{\delta \rightarrow 0} 0, \quad (43)$$

where \mathcal{H}_t^n is the σ -algebra generated by $\{\theta_1^n(r) - \theta_1^\infty(r) : r \leq t\}$.

The almost sure convergence is a direct consequence of Lemma 5.3. To prove the inequality, let \mathcal{H}_t be the σ -algebra generated by the random variables $\{\theta_1^n(r) : r \leq$

$t, n \in \mathbb{N}_*\}$. Since $\mathcal{H}_t^n \subseteq \mathcal{H}_t$, we can write:

$$\begin{aligned} & \mathbb{E}_\gamma [|(\theta_1^n(t+s) - \theta_1^\infty(t+s)) - (\theta_1^n(t) - \theta_1^\infty(t))| | \mathcal{H}_t^n] \\ & \leq \mathbb{E}_\gamma [\theta_1^n(t+s) - \theta_1^n(t) | \mathcal{H}_t^n] + \mathbb{E}_\gamma [\theta_1^\infty(t+s) - \theta_1^\infty(t) | \mathcal{H}_t^n] \\ & \leq \mathbb{E}_\gamma [\mathbb{E}_\gamma [\theta_1^n(t+s) - \theta_1^n(t) | \mathcal{H}_t] | \mathcal{H}_t^n] + \mathbb{E}_\gamma [\mathbb{E}_\gamma [\theta_1^\infty(t+s) - \theta_1^\infty(t) | \mathcal{H}_t] | \mathcal{H}_t^n]. \end{aligned} \quad (44)$$

Note that $\theta_1^n(t+s) - \theta_1^n(t)$ only depends on the values of $\theta_1^m(r)$ for $m \geq n$ and $r \leq t$ through the values of $\theta_1^n(r)$, $r \leq t$. So we can use Lemma 5.4 to compute the value of the first term:

$$\mathbb{E}_\gamma [\theta_1^n(t+s) - \theta_1^n(t) | \mathcal{H}_t] = \mathbb{E}_\gamma [\theta_1^n(s)] = s \sum_{x|_n} \bar{\gamma}_n(x|_n).$$

Since $\theta_1^n(t)$ and $\theta_1^n(t+s)$ converges in probability to $\theta_1^\infty(t)$ and $\theta_1^\infty(t+s)$ respectively (Proposition 5.2), we can take an increasing sequence n_m such that the convergence is almost sure. Then using Fatou's Lemma, we can conclude:

$$\mathbb{E}_\gamma [\theta_1^\infty(t+s) - \theta_1^\infty(t) | \mathcal{H}_t] \leq \liminf_{m \rightarrow \infty} \mathbb{E}_\gamma [\theta_1^{n_m}(t+s) - \theta_1^{n_m}(t) | \mathcal{H}_t] \leq \liminf_{m \rightarrow \infty} s \sum_{x|_{n_m}} \bar{\gamma}_{n_m}(x|_{n_m}).$$

Using the last results on (44) gives us (43). And completes the proof that θ_1^n converges in probability to θ_1^∞ uniformly in compacts.

$\theta_{k+1}^n - \theta_{k+1}^\infty$ converges in probability to the null function in the Skorohod topology if and only if for every $T > 0$:

$$\sup_{0 \leq t \leq T} |\theta_{k+1}^n(t) - \theta_{k+1}^\infty(t)| \xrightarrow[n \rightarrow \infty]{P} 0. \quad (45)$$

For each $x|_k \in \mathbb{N}_*^k$ fixed, define $\theta_{x|_k}^\infty$ from θ_{k+1}^∞ in an analogous way as done in Definition 4.2. Using Remark 4.1 and the last case, we can conclude that $\theta_{x|_k}^n - \theta_{x|_k}^\infty$ converges to the null function in probability in the Skorohod topology as $n \rightarrow \infty$.

Take $V_1 \subset V_2 \subset \dots \subset \mathbb{N}_*^k$ any increasing sequence of sets from \mathbb{N}_*^k such that $\bigcup_{l=1}^\infty V_l = \mathbb{N}_*^k$ and each V_l is finite. For an arbitrary $\epsilon > 0$, fix an l such that

$$\mathbb{P}_\gamma \left(\sum_{x|_k \notin V_l} \theta_{x|_k}^\infty(L(x|_k, T)) > \epsilon \right) < \epsilon.$$

Thence,

$$\begin{aligned}
\sup_{0 \leq t \leq T} |\theta_{k+1}^n(t) - \theta_{k+1}^\infty(t)| &= \sup_{0 \leq t \leq T} \left| \sum_{x|_k} \theta_{x|_k}^n(L(x|_k, t)) - \theta_{x|_k}^\infty(L(x|_k, t)) \right| \\
&\leq \sup_{0 \leq t \leq T} \sum_{x|_k \in V_l} |\theta_{x|_k}^n(L(x|_k, t)) - \theta_{x|_k}^\infty(L(x|_k, t))| + \sum_{x|_k \notin V_l} \theta_{x|_k}^n(L(x|_k, T)) \\
&\quad + \sum_{x|_k \notin V_l} \theta_{x|_k}^\infty(L(x|_k, T)) \\
&\leq \sup_{0 \leq t \leq T} \sum_{x|_k \in V_l} |\theta_{x|_k}^n(t) - \theta_{x|_k}^\infty(t)| + \sum_{x|_k \notin V_l} \theta_{x|_k}^n(L(x|_k, T)) + \sum_{x|_k \notin V_l} \theta_{x|_k}^\infty(L(x|_k, T)).
\end{aligned}$$

The first term converges to 0 in probability as $n \rightarrow \infty$ because V_l is finite. The third term is controlled by our choice of ϵ . Using an analogous argument as in (41), we can show that the second term converges to the third in probability as $n \rightarrow \infty$. \square

Remark 5.8. Using analogous arguments, we can prove Theorem 5.6 with $\tilde{\theta}_k^n$ in the place of θ_k^n . The only significant change in the proof comes from the equality $\mathbb{E}_\gamma[\tilde{\theta}_1^n(t)] = tW(\emptyset)$. However this actually slightly simplifies the proof.

Corollary 5.9. If $\sum_i (1 - \alpha_i) < \infty$, then for every $j, k \in \mathbb{N}_*$ with $j < k$ a.s.:

$$\theta_j^\infty = \theta_k^\infty \circ \theta_j^{k-1}.$$

Proof. Since, by Theorem 5.6, θ_j^n and θ_k^n converge in probability in the uniform norm to θ_j^∞ and θ_k^∞ respectively, we can take an increasing sequence n_m such that this convergence is almost sure. Fixing such a sequence, we can write that for any $T > 0$:

$$\begin{aligned}
&\sup_{t \in [0, T]} |\theta_j^\infty(t) - \theta_k^\infty(\theta_j^{k-1}(t))| \\
&\leq \sup_{t \in [0, T]} |\theta_j^\infty(t) - \theta_j^{n_m}(t)| + \sup_{t \in [0, T]} |\theta_j^{n_m}(\theta_j^{k-1}(t)) - \theta_k^\infty(\theta_j^{k-1}(t))| \\
&\xrightarrow[m.s.]{m \rightarrow \infty} 0.
\end{aligned}
\quad \square$$

Corollary 5.10. Suppose that $\sum_i (1 - \alpha_i) < \infty$. For any fixed $k \in \mathbb{N}_*$, if $s \geq 0$ is a discontinuity point of θ_k^∞ , then $s = \sigma_i^{k, x}$ for some $k, x \in \mathbb{N}_*$.

Proof. Using Theorem 5.6, we can take an increasing sequence $(n_m)_m$ such that, for an $T > s$ fixed arbitrarily:

$$\sup_{t \in [0, T]} |\theta_k^{n_m}(t) - \theta_k^\infty(t)| \xrightarrow[m.s.]{m \rightarrow \infty} 0.$$

Since s is a discontinuity point of θ_k^∞ and this is an non-decreasing càdlàg function, there exists an $\epsilon > 0$ such that $\theta_k^\infty(s) - \theta_k^\infty(s-) > \epsilon$, and this implies that $\theta_k^\infty(s) - \theta_k^\infty(s-h) > \epsilon$ for every $h \in (0, s)$.

For an arbitrary $h \in (0, s)$ we can write:

$$\begin{aligned}\theta_k^{n_m}(s) - \theta_k^{n_m}(s-h) &= \theta_k^{n_m}(s) - \theta_k^\infty(s) \\ &\quad + \theta_k^\infty(s) - \theta_k^\infty(s-h) \\ &\quad + \theta_k^\infty(s-h) - \theta_k^{n_m}(s-h).\end{aligned}$$

The first and third terms of this last equation converges *a.s.* to zero uniformly in h as $m \rightarrow \infty$, while the second term is always greater than ϵ . Therefore we conclude that, $\inf_{h \in (0, s)} \theta_k^{n_m}(s) - \theta_k^{n_m}(s-h) > \epsilon/2$ for large enough m , so s is a discontinuity point of $\theta_k^{n_m}$. Finally Remark 2.5 states that the set of discontinuity points of $\theta_k^{n_m}$ is $\{\sigma_i^{k,x} : i, x \in \mathbb{N}_*\}$. \square

Corollary 5.11. *Under the non-triviality assumption (22), for any $k \in \mathbb{N}_*$:*

$$\theta_k^\infty(\mathbb{R}) = \{t \geq 0 : Y_k(t) = \infty\} \text{ a.s.}$$

Proof. Start by taking $t \in \theta_k^\infty(\mathbb{R})$ and an $s \geq 0$ such that $\theta_k^\infty(s) = t$. Let us assume for contradiction hypothesis that $Y_k(t) = x < \infty$.

By definition, since $Y_k(t) = x$, there exists an $i \in \mathbb{N}_*$ such that $\theta_k^\infty(\sigma_i^{k,x}-) \leq t < \theta_k^\infty(\sigma_i^{k,x})$.

Since $t = \theta_k^\infty(s)$ and θ_k^∞ is strictly increasing (Theorem 4.5), the right inequality implies that $s < \sigma_i^{k,x}$, which in turn implies that $t = \theta_k^\infty(s) < \theta_k^\infty(\sigma_i^{k,x}-)$, which contradicts the first inequality from the last paragraph. Therefore $Y_k(t) = \infty$.

Let us assume that $t > 0$ is such that $Y_k(t) = \infty$. Take $s = \inf \{r > 0 : \theta_k^\infty(r) > t\}$. Using the right continuity of θ_k^∞ , we conclude that $\theta_k^\infty(s) \geq t$. Let us now assume, as a contradiction hypothesis, that $\theta_k^\infty(s) > t$. By the definition of s , we know that $\theta_k^\infty(s-) \leq t$. Therefore $\theta_k^\infty(s-) \neq \theta_k^\infty(s)$.

Using Corollary 5.10, we obtain that $s = \sigma_i^{k,x}$ for some $i, x \in \mathbb{N}_*$. Therefore $\theta_k^\infty(\sigma_i^{k,x}-) \leq t < \theta_k^\infty(\sigma_i^{k,x})$, which implies that $Y_k(t) = x < \infty$, contradicting our choice for t . \square

Remark 5.12. *If $\sum_i (1 - \alpha_i) < \infty$, Corollary 5.9 and 5.11 readily imply that:*

$$\{t \geq 0 : Y_k(t) = \infty\} \subseteq \{t \geq 0 : Y_{k+1}(t) = \infty\}$$

almost surely for any $k \in \mathbb{N}_$.*

Corollary 5.13. *Under the non-triviality condition (22), the set $\{t \geq 0 : Y_k(t) = \infty\}$ has null Lebesgue measure a.s. for every $k \in \mathbb{N}_*$.*

Proof. We will prove that the following equality is valid for every $t > 0$ almost surely:

$$\theta_k^\infty(t) = \sum_{x=1}^{\infty} \sum_{i=1}^{N^{k,x}(t)} \theta_k^\infty(\sigma_i^{k,x}) - \theta_k^\infty(\sigma_i^{k,x}-). \quad (46)$$

With this we are showing that θ_k^∞ is a step function, therefore its image has null Lebesgue measure. The result then follows from Corollary 5.11.

Note that (46) is not a direct consequence of the uniform convergence in Theorem 5.6. It is possible to construct a sequence of step functions (f_n) that converge uniformly to another function f , all having exactly the same discontinuities but f itself is not a step function.

For the rest of the proof, let us fix $\underline{\gamma}$ and prove the result for almost every such choice.

Since θ_k^∞ is a non-decreasing, càdlàg function, it is the distribution function of a measure. The right hand side of (46) can be interpreted as the sum over some points of this measure. Therefore we conclude that:

$$\theta_k^\infty(t) \geq \sum_{x=1}^{\infty} \sum_{i=1}^{N^{k,x}(t)} \theta_k^\infty(\sigma_i^{k,x}) - \theta_k^\infty(\sigma_i^{k,x}-).$$

To show the reverse inequality, using Remark 5.8, let us take an increasing sequence $(n_m)_m$ such that this convergence is almost sure. We note that for any fixed $N \in \mathbb{N}_*$:

$$\begin{aligned} \theta_k^\infty(t) &= \lim_{m \rightarrow \infty} \tilde{\theta}_k^{n_m}(t) \\ &= \lim_{m \rightarrow \infty} \sum_{x=1}^{\infty} \sum_{i=1}^{N^{k,x}(t)} \tilde{\theta}_k^{n_m}(\sigma_i^{k,x}) - \tilde{\theta}_k^{n_m}(\sigma_i^{k,x}-) \\ &\leq \limsup_{m \rightarrow \infty} \sum_{x=1}^N \sum_{i=1}^{N^{k,x}(t)} \tilde{\theta}_k^{n_m}(\sigma_i^{k,x}) - \tilde{\theta}_k^{n_m}(\sigma_i^{k,x}-) \end{aligned} \quad (47)$$

$$+ \limsup_{m \rightarrow \infty} \sum_{x=N+1}^{\infty} \sum_{i=1}^{N^{k,x}(t)} \tilde{\theta}_k^{n_m}(\sigma_i^{k,x}) - \tilde{\theta}_k^{n_m}(\sigma_i^{k,x}-). \quad (48)$$

We also note that (47) is equal to:

$$\sum_{x=1}^N \sum_{i=1}^{N^{k,x}(t)} \theta_k^\infty(\sigma_i^{k,x}) - \theta_k^\infty(\sigma_i^{k,x}-) \xrightarrow[a.s.]{N \rightarrow \infty} \sum_{x=1}^{\infty} \sum_{i=1}^{N^{k,x}(t)} \theta_k^\infty(\sigma_i^{k,x}) - \theta_k^\infty(\sigma_i^{k,x}-).$$

To complete the proof, we need to show that (48) converges to zero in probability as $N \rightarrow \infty$. By analogous arguments as those used in the proof of Theorem 4.4, $(\tilde{\theta}_k^n(\sigma_i^{k,x}) - \tilde{\theta}_k^n(\sigma_i^{k,x}-))_n$ is appropriately a martingale. Therefore the sequence that we are taking the limsup of in (48) is a positive martingale. Using again a martingale convergence theorem, we conclude that the limsup in that expression is in fact a limit.

Denoting (48) by K_N and again letting \mathcal{G}_k be the σ -algebra generated by all dynamical information up to the level k , we can use Fatou's Lemma to obtain that:

$$\begin{aligned}
\mathbb{E}_\gamma[K_N] &\leq \liminf_{m \rightarrow \infty} \mathbb{E}_\gamma \left[\sum_{x=N+1}^{\infty} \sum_{i=1}^{N^{k,x}(t)} \tilde{\theta}_k^{n_m}(\sigma_i^{k,x}) - \tilde{\theta}_k^{n_m}(\sigma_i^{k,x}-) \right] \\
&= \liminf_{m \rightarrow \infty} \sum_{x=N+1}^{\infty} \mathbb{E}_\gamma \left[\sum_{i=1}^{N^{k,x}(t)} \mathbb{E}_\gamma \left[\tilde{\theta}_k^{n_m}(\sigma_i^{k,x}) - \tilde{\theta}_k^{n_m}(\sigma_i^{k,x}-) \middle| \mathcal{F}_{k-1} \right] \right] \\
&= \liminf_{m \rightarrow \infty} \sum_{x=N+1}^{\infty} \mathbb{E}_\gamma \left[\sum_{i=1}^{N^{k,x}(t)} \gamma_k(X_{k-1}(\sigma_i^{k,x})x) W(X_{k-1}(\sigma_i^{k,x})x) \right] \\
&= \sum_{x=N+1}^{\infty} \mathbb{E}_\gamma \left[\sum_{i=1}^{N^{k,x}(t)} \gamma_k(X_{k-1}(\sigma_i^{k,x})x) W(X_{k-1}(\sigma_i^{k,x})x) \right] \\
&\leq \mathbb{E}_\gamma \left[\sum_{x=1}^{\infty} \sum_{i=1}^{N^{k,x}(t)} \gamma_k(X_{k-1}(\sigma_i^{k,x})x) W(X_{k-1}(\sigma_i^{k,x})x) \right] \\
&= \mathbb{E}_\gamma [\tilde{\Xi}_k(t)] < \infty.
\end{aligned} \tag{49}$$

So we proved that the sum in (49) is convergent and therefore converges to zero as $N \rightarrow \infty$. Finally K_N converges in L_1 to zero and therefore in probability as well. \square

Finally we can prove the main result of this paper, stated on Section 2.

Proof of Theorem 2.7. We will omit the proof that \mathbb{Y} is a càdlàg process, since the argument is quite similar to the ones used to prove the convergence.

Take any increasing sequence (b_n) of natural numbers. We will show that this sequence has a subsequence $(k_n)_n$ over which X_{k_n} converges in probability to \mathbb{Y} as $n \rightarrow \infty$. This implies that X_k converges in probability to \mathbb{Y} .

Theorem 5.6 guarantees that for each j there exists a subsequence (a_n) of (b_n) such that $\theta_j^{a_n} - \theta_j^\infty$ converges almost surely in the uniform norm. Using Cantor's diagonal method we can show that there exists (k_n) a subsequence of (a_n) such that $\theta_j^{k_n} - \theta_j^\infty$

converges almost surely for all j . Fix such a subsequence. It is along it that we will show the convergence.

Fix a realization of the process, we will show that for almost all such realizations and for every $T > 0$, there exists a sequence (λ_n) of functions such that $\lambda_n : [0, \infty) \rightarrow [0, \infty)$ is strictly increasing Lipschitz continuous, that satisfies:

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \rho(X_{k_n}(t), \mathbb{Y}(\lambda_n(t))) = 0, \quad (50)$$

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} |\lambda_n(t) - t| = 0. \quad (51)$$

Theorem 5.3 from Chapter 3 of [15] guarantees that showing this is equivalent to showing that X_{k_n} converges to \mathbb{Y} almost surely in the Skorohod topology.

We will show that for every $\epsilon > 0$ there exists a sequence (λ_n) such that (51) is satisfied and the quantity in (50) is smaller than ϵ . This will imply that exists a sequence (λ_n) that satisfies (50) and (51).

For a fixed $\epsilon > 0$, take $N, M \in \mathbb{N}_*$ such that $\sum_{j \geq N} \frac{1}{2^j} < \epsilon/2$ and $1/(M+1) < \epsilon/2$, and assume that n is large enough so that $k_n > N$.

To construct λ_n , for $k_n > N$, take $S_n := \{\theta_j^N(\sigma_i^{j,x}-), \theta_j^N(\sigma_i^{j,x}) : x \leq M, j < N, \theta_j^{k_n}(\sigma_i^{j,x}-) \leq T\}$. Define λ_n such that for every $s \in S_n$:

$$\lambda_n(\theta_{N+1}^{k_n}(s)) = \theta_{N+1}^\infty(s).$$

Complete λ_n linearly between these points and let it evolve linearly with angular coefficient 1 after the last point. From the facts that S_n is finite and λ_n is linear by parts, it readily follows that λ_n is Lipschitz continuous. Theorem 4.5 guarantees that λ_n is strictly increasing, so it qualifies as a candidate for temporal distortion.

Using Remark 2.4, we know that the j -th coordinate, $j < N$, of $X_{k_n}(t)$ is equal to an $x \leq M$ if and only if:

$$\begin{aligned} t \in \bigcup_{i=1} [\theta_j^{k_n}(\sigma_i^{j,x}-), \theta_j^{k_n}(\sigma_i^{j,x})] &= \bigcup_{i=1} [\theta_{N+1}^{k_n}(\theta_j^N(\sigma_i^{j,x}-)), \theta_{N+1}^{k_n}(\theta_j^N(\sigma_i^{j,x}))] \Leftrightarrow \\ \lambda_n(t) \in \bigcup_{i=1} [\theta_{N+1}^\infty(\theta_j^N(\sigma_i^{j,x}-)), \theta_{N+1}^\infty(\theta_j^N(\sigma_i^{j,x}))] &= \bigcup_{i=1} [\theta_j^\infty(\sigma_i^{j,x}-), \theta_{N+1}^\infty(\sigma_i^{j,x})]. \end{aligned}$$

Note that we have used Corollary 5.9 in the last passage. With this we conclude that, for any coordinate $j < N$:

$$\begin{aligned} \sup_{t \in [0, T]} \rho_0(X_{k_n, j}(t), Y_j(\lambda_n(t))) &\leq \frac{1}{M+1} < \frac{\epsilon}{2}, \\ \sup_{t \in [0, T]} \rho(X_{k_n}(t), \mathbb{Y}(\lambda_n(t))) &< \frac{\epsilon}{2} + \sum_{j \geq N} \frac{1}{2^j} < \epsilon. \end{aligned}$$

This concludes (50). To show (51) take $T' > 0$ such that $\theta_j^\infty(T') > T + 1$ for every $j < N$, and note that, for large enough n :

$$\sup_{t \in [0, T]} |\lambda_n(t) - t| = \max_{s \in S_n} |\theta_{N+1}^{k_n}(s) - \theta_{N+1}^\infty(s)| \leq \sup_{s \in [0, T']} |\theta_{N+1}^{k_n}(s) - \theta_{N+1}^\infty(s)| \xrightarrow{n \rightarrow \infty} 0. \quad \square$$

Remark 5.14. *We come back to the conditions (1) and (6) to discuss what if any of them is missing. Without (1), we cannot insure the existence of $W(\cdot)$ and thus of the θ_k^∞ 's, without which we cannot define the limiting K process. One might try to obtain convergence in distribution of the clocks, but we do not see an approach to, say, take the limit of the Laplace transform of θ_k^n as $n \rightarrow \infty$. If we keep (1) but try to relax (6), then it is the nontriviality of the θ_k^∞ 's that is at stake: we know or presume that it is identically zero in this case, and this disables the definition of a meaningful limiting K process. We would need in this case to rescale θ_k^n before taking the limit, but it is not clear to us even which would be the right scale be, or if this would lead to the definition of the correct limit for the K process. If this could be done, the limiting process could be quite different from the one we obtained above.*

6 Empirical Measure

In this section we will assume that $\underline{\gamma}$ is fixed. All results from this section are valid for almost all choices for this random environment (under nontriviality conditions).

The main result of this section is the computation of the asymptotic empirical measure of the K process on a tree with infinite depth, that is, the proportion of the time that this process spends on cylinders $[x|_k] = \{y|_\infty \in \bar{\mathbb{N}}_*^{\mathbb{N}^*} : y|_k = x|_k\}$. We will show that it is almost surely given by:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \mathbb{I} \{Y(t) \in [x|_k]\} dt = \frac{\bar{\gamma}_k(x|_k) \mathbb{E}_\gamma [\theta_{x|_k}^\infty(1)]}{\mathbb{E}_\gamma [\theta_1^\infty(1)]}. \quad (52)$$

We showed that the right hand side of this last expression is well-defined *a.s.* in Theorem 4.4 (since in particular we are under nontriviality conditions). To compute the empirical measure, we will rely on the following assumption.

Assumption 1. *The K process on a tree with finite depth is strongly Markovian.*

A proof of the Markov property of the finite level K process can be found in [17]. The Feller property (as well as the Markov property itself) should follow from arguments similar to those for the 1 level case used in [8], establishing the strong Markov property. We choose to not go to detail, and leave the issue as an assumption.

Before computing the empirical measure, let us prove some auxiliary results.

Proposition 6.1. *If T is a positive random variable independent from $\theta_{x|_k}^\infty$, then*

$$\mathbb{E}_\gamma [\theta_{x|_k}^\infty(T)] = \mathbb{E}_\gamma [\theta_{x|_k}^\infty(1)] \mathbb{E}_\gamma(T).$$

Proof. We will prove this result only for θ_1^∞ , Remark 4.1 extends the result to $\theta_{x|_k}^\infty$.

Fix an arbitrary $n, m \in \mathbb{N}_*$, since θ_1^∞ is a subordinator (Remark 5.5) then $\theta_1^\infty(n/m)$ has the same law then the sum of n independent copies of $\theta_1^\infty(1/m)$. By the same argument $\theta_1^\infty(1)$ has the same law as the sum of m independent copies of $\theta_1^\infty(1/m)$. Therefore:

$$\mathbb{E}_\gamma \left[\theta_1^\infty \left(\frac{n}{m} \right) \right] = n \mathbb{E}_\gamma \left[\theta_1^\infty \left(\frac{1}{m} \right) \right] = \frac{n}{m} \mathbb{E}_\gamma [\theta_1^\infty(1)]$$

For an arbitrary real $t > 0$, take q, r rational numbers such that $0 < q < t < r$. Since a subordinator is monotonic then:

$$\begin{aligned} \mathbb{E}_\gamma [\theta_1^\infty(q)] &\leq \mathbb{E}_\gamma [\theta_1^\infty(t)] \leq \mathbb{E}_\gamma [\theta_1^\infty(r)] \\ \Leftrightarrow q \mathbb{E}_\gamma [\theta_1^\infty(1)] &\leq \mathbb{E}_\gamma [\theta_1^\infty(t)] \leq r \mathbb{E}_\gamma [\theta_1^\infty(1)]. \end{aligned}$$

Since q, r were taken arbitrarily, we have that $\mathbb{E}_\gamma [\theta_1^\infty(t)] = t \mathbb{E}_\gamma [\theta_1^\infty(1)]$. To complete the proof, let ν be the probability measure associated with T . Since T is independent from θ_1^∞ , we conclude that

$$\mathbb{E}_\gamma [\theta_1^\infty(T)] = \int \mathbb{E}_\gamma [\theta_1^\infty(t)] \nu(dt) = \mathbb{E}_\gamma [\theta_1^\infty(1)] \int t \nu(dt) = \mathbb{E}_\gamma [\theta_1^\infty(1)] \mathbb{E}_\gamma(T).$$

□

The next result states that, for a fixed $t \geq 0$, the family $\{\mathbb{E}_\gamma[\theta_{x|_k}^\infty(t)] : k \in \mathbb{N}_*, x|_k \in \mathbb{N}_*^k\}$ obeys a composition law analogous as the one stated in Proposition 3.5 for the family $\{W(x|_k) : k \in \mathbb{N}_*, x|_k \in \mathbb{N}_*^k\}$.

Proposition 6.2. *For any fixed $t > 0$ and $x|_k \in \mathbb{N}_*^k$, if $\sum_i (1 - \alpha_i) < \infty$:*

$$\mathbb{E}_\gamma [\theta_{x|_k}^\infty(t)] = \sum_{x_{k+1}} \gamma_{k+1}(x|_{k+1}) \mathbb{E}_\gamma [\theta_{x|_{k+1}}^\infty(t)]. \quad (53)$$

Furthermore:

$$\mathbb{E}_\gamma \left[\theta_{x|_k}^\infty(\sigma_1^{k+1, x_{k+1}} -) \right] = \mathbb{E}_\gamma [\theta_{x|_k}^\infty(1)] - \gamma_{k+1}(x|_{k+1}) \mathbb{E}_\gamma [\theta_{x|_{k+1}}^\infty(1)]. \quad (54)$$

Sketch of proof. To prove (53), apply Corollary 5.9 to break the contribution from each x_{k+1} to $\theta_{x|_k}^\infty$, and then use Proposition 6.1.

To prove (54), note that $\theta_{x|_k}^\infty$, up to time $\sigma_1^{k+1, x_{k+1}}$ is independent from this time, with the exception that it does not see any point of the Poisson Process $N^{k+1, x_{k+1}}$, which would be equivalent of setting $\gamma_{k+1}(x|_{k+1}) = 0$. Denoting by $\widehat{\theta}_{x|_k}^\infty$ a version of $\theta_{x|_k}^\infty$ in which this modification was made, we can use the previous result and Proposition 6.1 to obtain:

$$\begin{aligned} \mathbb{E}_\gamma \left[\theta_{x|_k}^\infty(\sigma_1^{k+1, x_{k+1}} -) \right] &= \mathbb{E}_\gamma \left[\widehat{\theta}_{x|_k}^\infty(\sigma_1^{k+1, x_{k+1}}) \right] \\ &= \mathbb{E}_\gamma(\sigma_1^{k+1, x_{k+1}}) \mathbb{E}_\gamma \left[\widehat{\theta}_{x|_k}^\infty(1) \right] \\ &= \sum_{y \neq x_{k+1}} \gamma_{k+1}(x|_k y) \mathbb{E}_\gamma \left[\theta_{x|_k y}^\infty(1) \right] \\ &= \mathbb{E}_\gamma \left[\theta_{x|_k}^\infty(1) \right] - \gamma_{k+1}(x|_{k+1}) \mathbb{E}_\gamma \left[\theta_{x|_{k+1}}^\infty(1) \right] \quad \square \end{aligned}$$

Definition 6.1. Let us denote the first k coordinates of the process \mathbb{Y} by $Y|_k$, that is:

$$Y|_k := (Y_1, \dots, Y_k)$$

Definition 6.2. For a fixed $y|_k \in \mathbb{N}_*^k$, let us denote by U_i and V_i the i -th entrance and exit times respectively of \mathbb{Y} in $[y|_k]$. That is, we define $V_0 := 0$ and for $i = 1, 2, \dots$:

$$U_i := \inf \{t > V_{i-1} : Y|_k(t) = y|_k\} \quad (55a)$$

$$V_i := \inf \{t > U_i : Y|_k(t) \neq y|_k\} \quad (55b)$$

Remark 6.3. Since \mathbb{Y} is right continuous, then $Y|_k(U_i) = y|_k$ and $Y|_k(V_i) \neq y|_k$. Furthermore it is true that $Y_j(U_i) = \infty$ for every $j > k$ and that:

$$Y_j(V_i) = \begin{cases} y_j & \text{if } j < k, \\ \infty & \text{otherwise.} \end{cases}$$

The increment $V_i - U_i$ is the time spent by \mathbb{Y} on $y|_k$ on its i -th visit. It is equal to $\theta_k^\infty(\sigma_j^{k, y_k}) - \theta_k^\infty(\sigma_j^{k, y_k} -)$ for some j . Therefore $\mathbb{E}_\gamma(V_i - U_i) = \mathbb{E}_\gamma \left[\theta_{y|_k}^\infty \left(\gamma_k(x|_k) T_1^{k, y_k} \right) \right] = \gamma_k(y|_k) \mathbb{E}_\gamma(\theta_{y|_k}^\infty(1))$.

Proposition 6.4. The increment $U_{i+1} - V_i$ is the time spent outside of $[y|_k]$ between successive visits to this cylinder. Its expected value can be computed as:

$$\mathbb{E}_\gamma(U_{i+1} - V_i) = \frac{\mathbb{E}_\gamma(\theta_1^\infty(1))}{\bar{\gamma}_{k-1}(y|_{k-1})} - \gamma_k(y|_k) \mathbb{E}_\gamma(\theta_{y|_k}^\infty(1)), \quad (56)$$

under the convention that $\bar{\gamma}_0(y|_0) := 1$.

Proof. Let us denote by S_j^1, S_j^2, \dots the times between visits to $y|_j$. We claim that it follows from Assumption 1 that this variables form an *i.i.d.* sequence. Indeed we can write the cycles determined by the successive visits of $Y|_j$ to $y|_j$ in terms of the cycles determined by the successive visits of X_j to $y|_j$, where X_j is the j level K process constructed in Section 2. Each former cycle is obtained as the sum over constancy intervals of X_j within the corresponding latter cycle of $\theta_{x|_j}^\infty(L_j(x|_j, b_j)) - \theta_{x|_j}^\infty(L_j(x|_j, a_j))$, where $x|_j$ is the constant value of X_j within the respective constancy interval $[a_j, b_j]$. The strong Markov property of X_j implies that the distribution of constancy intervals of X_j within the latter cycles are *i.i.d.* when we vary those cycles. Since moreover $\theta_{x|_j}^\infty(L_j(x|_j, b_j)) - \theta_{x|_j}^\infty(L_j(x|_j, a_j))$ are independent when we vary the constancy intervals (see Remark 4.1 and Lemma 5.4 above), the claim follows. (One ought also to be able to make an argument for the claim, dispensing with Assumption 1, by using directly the structure of the model together with the lack of memory of the exponential distribution.)

Let us now define $a_j := \mathbb{E}_\gamma(S_j^1)$. We want to compute a_k .

At time V_i , the process just exited $y|_k$, that is, $V_i = \theta_k^\infty(\sigma_{i'}^{k, y_k})$ for some i' . There exists an $\sigma_{i''}^{k-1, y_{k-1}}$ such that $\sigma_{i'}^{k, y_k} \in [\Xi_{k-1}(\sigma_{i''}^{k-1, y_{k-1}} -), \Xi_{k-1}(\sigma_{i''}^{k-1, y_{k-1}})]$. This interval has length $\gamma_{k-1}(y|_{k-1})T_{i''}^{k-1, y_{k-1}}$. Because of the loss of memory of the exponential distribution, the distribution of $\Xi_{k-1}(\sigma_{i''}^{k-1, y_{k-1}}) - \sigma_{i'}^{k, y_k}$ is exactly the same as the distribution of the whole interval.

With probability $p = \gamma_{k-1}(y|_{k-1})/(1 + \gamma_{k-1}(y|_{k-1}))$ it will happen that $\sigma_{i'+1}^{k, y_k} < \Xi_{k-1}(\sigma_{i''}^{k-1, y_{k-1}})$. In this case the K process will visit $y|_k$ again before exiting $y|_{k-1}$.

If this does not happen it will take a time S_{k-1}^1 for the process to visit $y|_{k-1}$ again, after that it will have a probability p of visiting $y|_k$ during this visit, if this does not happen then it will take a time S_{k-1}^2 for a third try, and so on.

Therefore the time spent outside of $y|_{k-1}$ has the same law as $\sum_{i=1}^M S_{k-1}^i$, where M is a geometric random variable, with success probability p , independent of $S_{k-1}^i, i = 1, 2, \dots$

The total time spent on $y|_{k-1}$ but outside of $y|_k$ has the same law as $\theta_{y|_{k-1}}(\sigma_1^{k, y_k} -)$. Therefore:

$$\begin{aligned} a_k &= a_{k-1} \frac{1-p}{p} + \mathbb{E}_\gamma \left[\theta_{y|_{k-1}}(\sigma_1^{k, y_k} -) \right] \\ &= \frac{a_{k-1}}{\gamma_{k-1}(y|_{k-1})} + \mathbb{E}_\gamma \left[\theta_{y|_{k-1}}^\infty(1) \right] - \gamma_k(y|_k) \mathbb{E}_\gamma \left[\theta_{y|_k}^\infty(1) \right]. \end{aligned}$$

Applying an induction in k , we obtain (56). \square

It follows from Assumption 1, as argued above, that increments between (U_i, V_i) are independent from each other as i varies. Therefore we can use the strong law of large

numbers to compute the proportion of the time that the process stays on the state $y|_k$ as:

$$\begin{aligned}\pi(y|_k) &:= \lim_{n \rightarrow \infty} \frac{\sum_{i=1}^{\infty} V_i - U_i}{V_n} \stackrel{a.s.}{=} \frac{\mathbb{E}_{\gamma}[V_i - U_i]}{\mathbb{E}_{\gamma}[U_{i+1} - U_i]} \\ &= \frac{\mathbb{E}_{\gamma}[V_i - U_i]}{\mathbb{E}_{\gamma}[U_{i+1} - V_i] + \mathbb{E}_{\gamma}[V_i - U_i]} = \frac{\bar{\gamma}_k(y|_k) \mathbb{E}_{\gamma}[\theta_{y|_k}^{\infty}(1)]}{\mathbb{E}_{\gamma}[\theta_1^{\infty}(1)]},\end{aligned}$$

and formula (52) is established. Proposition 6.2 can be then used to verify the conditions of Kolmogorov's Consistency Theorem, concluding that this function π defines a probability measure on the product σ -algebra. It should be the equilibrium measure for the infinite level GREM-like K process.

We finish with the remark that in case the upper bound for $\mathbb{E}_{\gamma}[\theta_{x|_k}^{\infty}(1)]$ in Theorem 4.4 saturated (that would be the case if the convergence $\theta^n(1) \rightarrow \theta^{\infty}(1)$ took place in L_1), then π would have a more explicit, nicer looking form, namely

$$\pi(y|_k) = \frac{\bar{\gamma}_k(y|_k) W(y|_k)}{W(\emptyset)}.$$

At the moment, we do not know whether this is the case or not.

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